Minimal Equational Theories for Quantum Circuits

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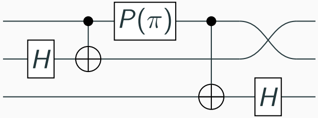
What is it all about?

Quantum circuits are a rigourous graphical language used to represent quantum algorithms.



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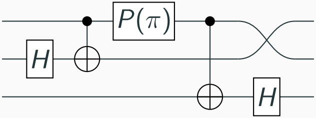

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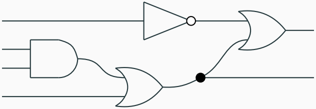
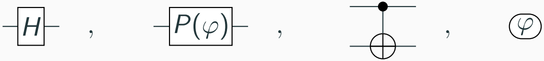


Quantum circuits as a graphical language

Quantum circuits are generated by the universal gateset



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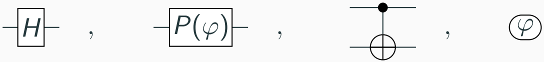

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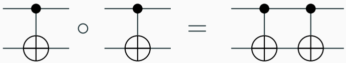

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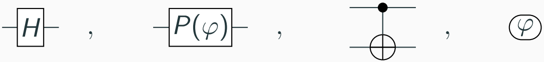

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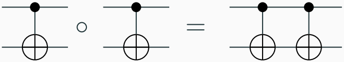

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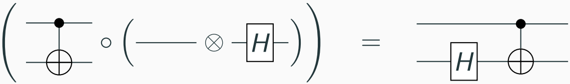


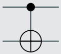
Standard interpretation of quantum circuits

Interpretation

$$[\![C_2 \circ C_1]\!] = [\![C_2]\!] \circ [\![C_1]\!] \qquad [\![C_1 \otimes C_2]\!] = [\![C_1]\!] \otimes [\![C_2]\!]$$

$$[\![\ \square\ ]\!] = (1) \qquad [\![\ \textcircled{\varphi}\ ]\!] = (e^{i\varphi})$$

$$[\![\ \text{---}\ ]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [\![\ -\boxed{H}-\ ]\!] = {}^1\!/\!\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [\![\ -\boxed{P(\varphi)}-\ ]\!] = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

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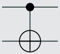
$[\![$  $]\!]$ $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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Definition "up to deformation"

Formally, quantum circuits are defined as a symmetric monoidal category, which ensure some deformation equations such that

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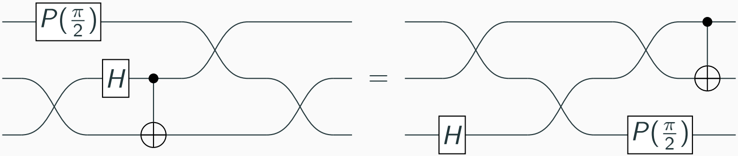


Other gates as shortcut notations

Other usual gates can be defined as shortcut notation by composition of the generators.

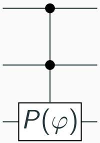
$$-\boxed{Z}- \;:=\; -\boxed{P(\pi)}-$$

$$-\boxed{X}- \;:=\; -\boxed{H}-\boxed{Z}-\boxed{H}-$$

$$-\boxed{R_X(\theta)}- \;:=\; \left(\,{}^{-\theta}\!/_{2}\,\right)\; -\boxed{H}-\boxed{P(\theta)}-\boxed{H}-$$

Controlled gates as shortcut notations

We use the standard bullet notation for controlled gates.



Controlled gates can be constructed inductively. The $(n+1)$-controlled gate is a shortcut containing several instances of n-controlled gates.

Note that unfolding the inductive definition divides the parameters by 2.

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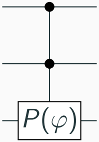

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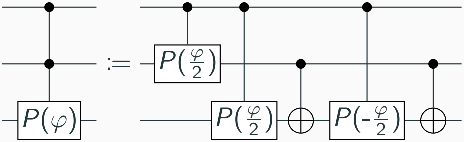

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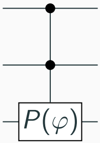

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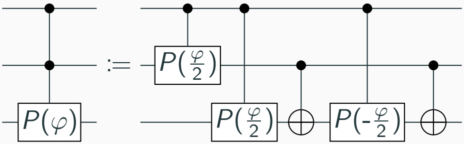

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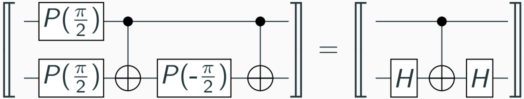
Motivations

Distinct circuits can have the same interpretation.



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Given a quantum algorithm, which circuit is the best?

Motivations:

- Resource optimisation (for instance the number of gates).
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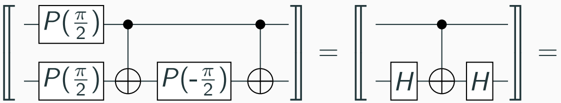

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Using equations to transform circuits

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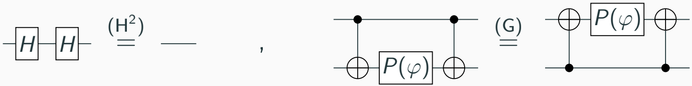

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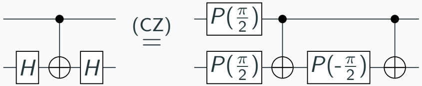

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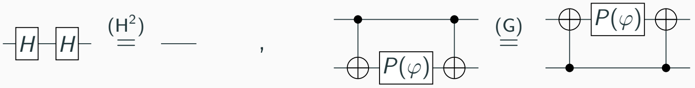

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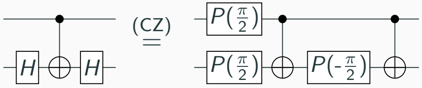

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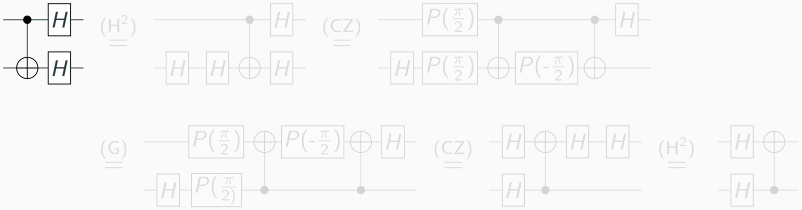

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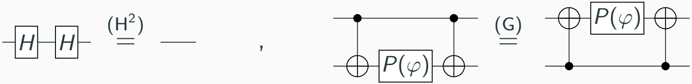

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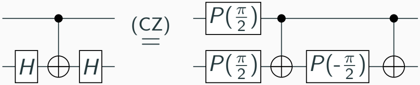

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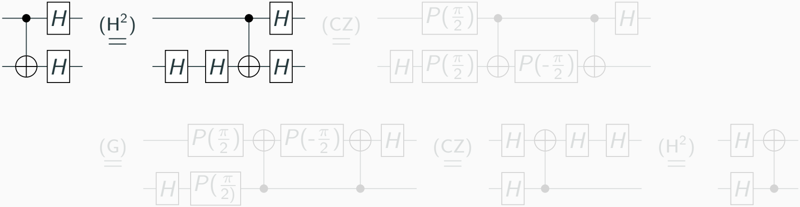

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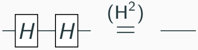

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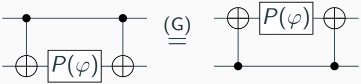

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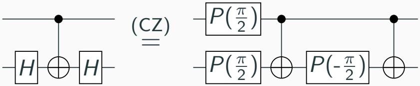

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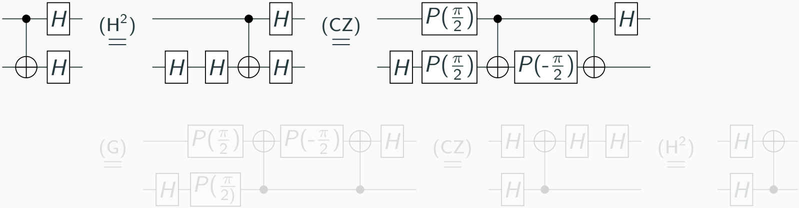

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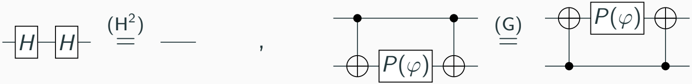

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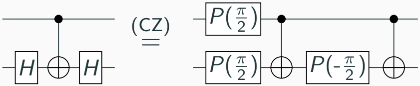

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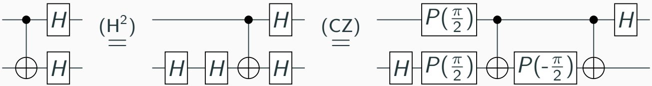

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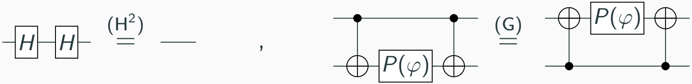

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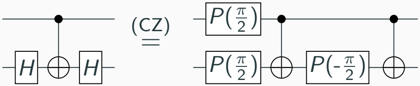

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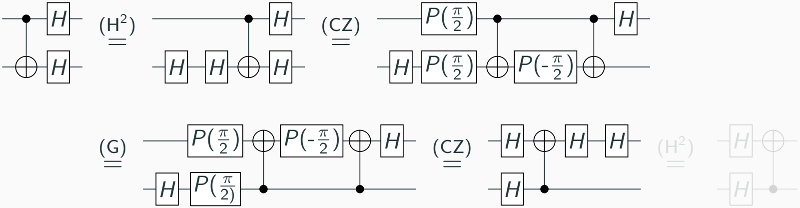

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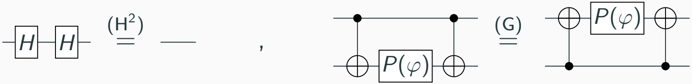

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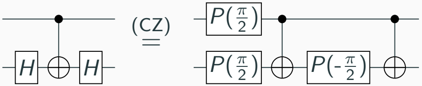

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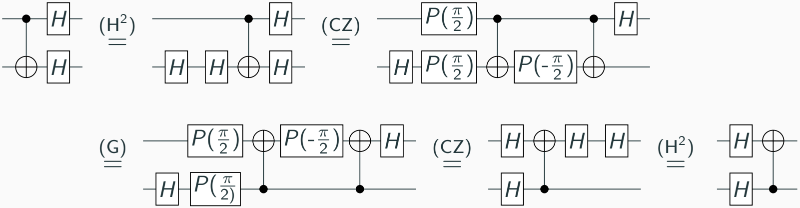

Soundness and completeness

Is there an equational theory (i.e. a set of axioms) Γ from which we can derive any true equation and only true equations?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \quad : \quad \Gamma \vdash C_1 = C_2 \implies [\![C_1]\!] = [\![C_2]\!]$$

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[Clément,Heurtel,Mansfield,Perdrix,Valiron'2023]

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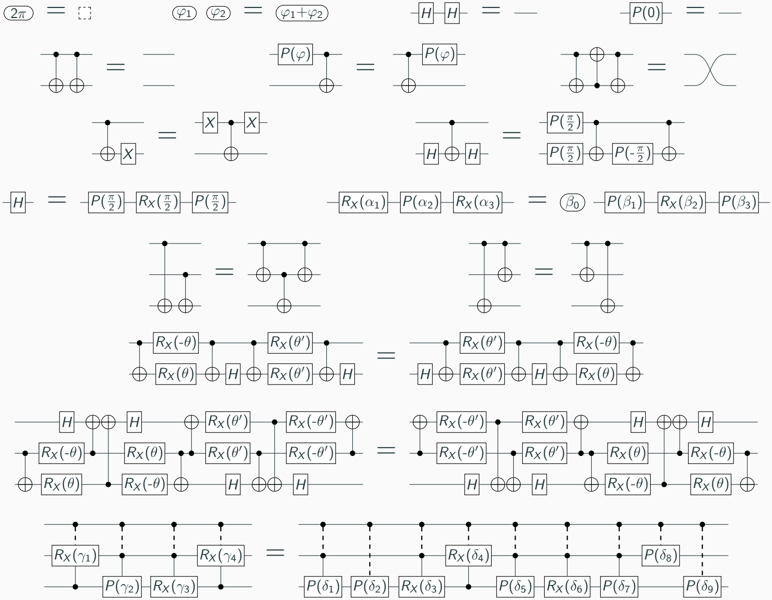
Complete and sound equational theory



Euler decomposition equation

This equation follows from the well-known Euler-decomposition which states that any unitary can be decomposed, up to a global phase, into basic X- and Z-rotations.

$$-\boxed{R_X(\alpha_1)}-\boxed{P(\alpha_2)}-\boxed{R_X(\alpha_3)}- \;=\; (\beta_0)\; -\boxed{P(\beta_1)}-\boxed{R_X(\beta_2)}-\boxed{P(\beta_3)}-$$

It represents a family of equations: there are explicit trigonometric relations to compute $\beta_0, \beta_1, \beta_2, \beta_3$ as functions of $\alpha_1, \alpha_2, \alpha_3$.

By choosing specific parameters, we can retrieve simple equations, such that

$$-\boxed{P(\varphi_1)}-\boxed{P(\varphi_2)}- \;=\; -\boxed{P(\varphi_1+\varphi_2)}- \qquad -\boxed{X}-\boxed{P(\varphi)}-\boxed{X}- \;=\; (\varphi)\; -\boxed{P(-\varphi)}-$$

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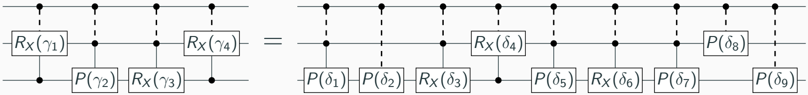
The weird equation



Similarly to the Euler decomposition equation, it represents a family of equations: there is an instance of this equation in the equational theory for any number of wires $n \geq 2$ and for any parameters $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}$.

The presence of such weird equation is the consequence of the technique used to prove completeness: the proof is based on back and forth translations between quantum circuits and optical circuits.

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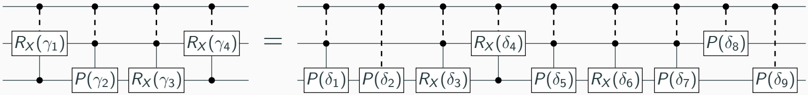

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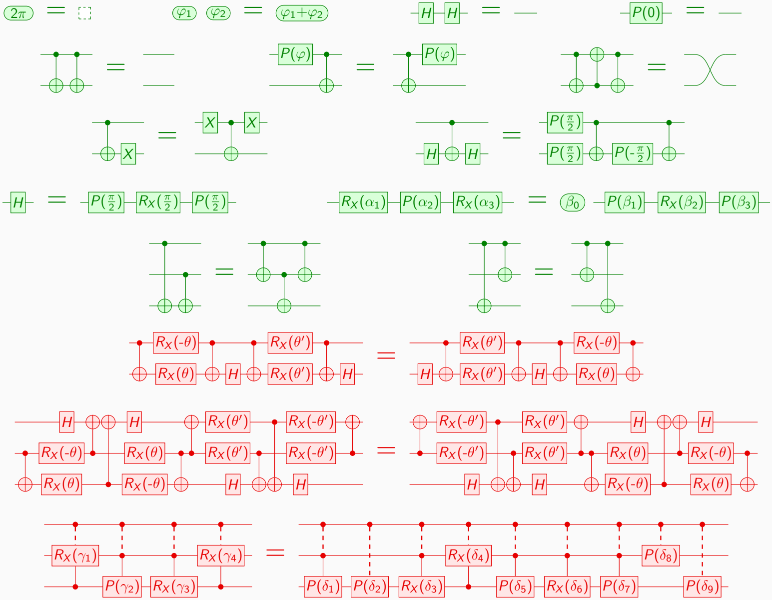
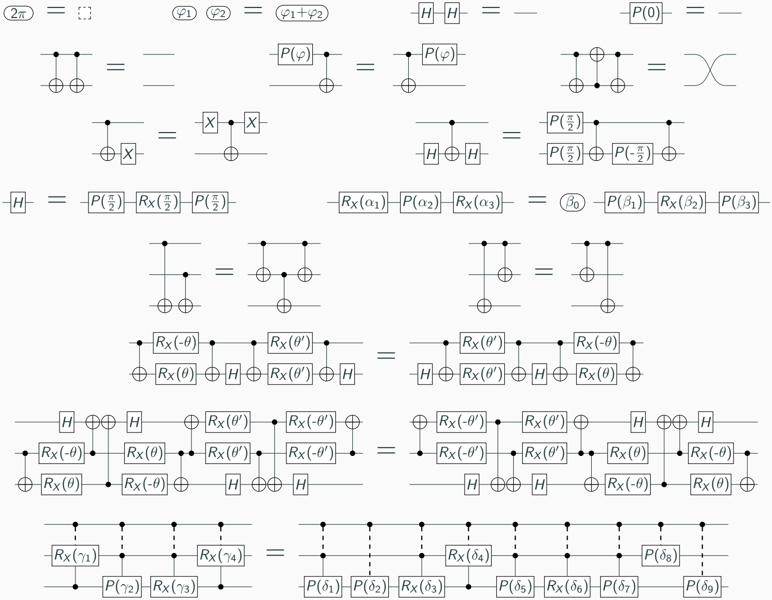
Some easy and some intricate equations



Simplifications



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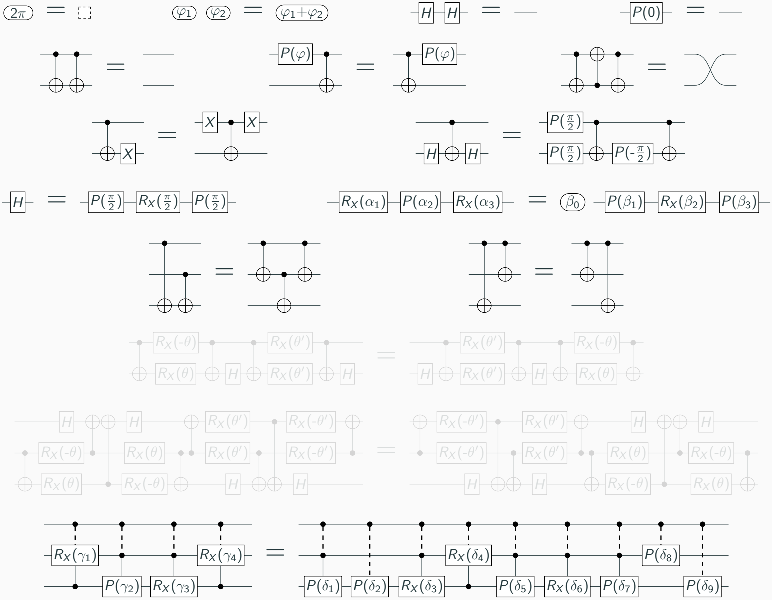

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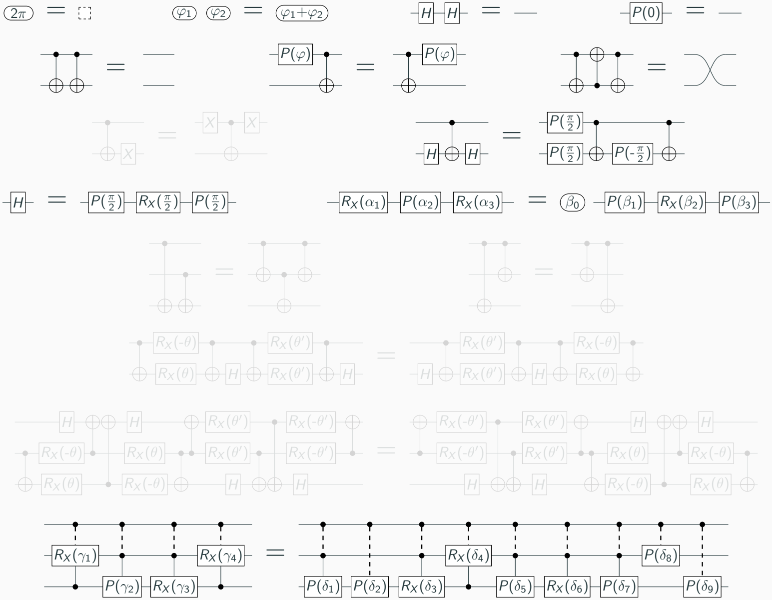

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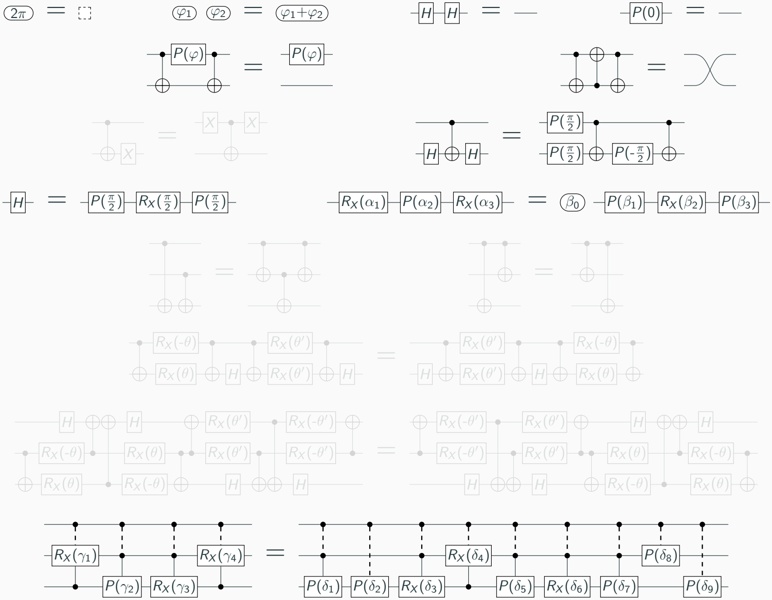

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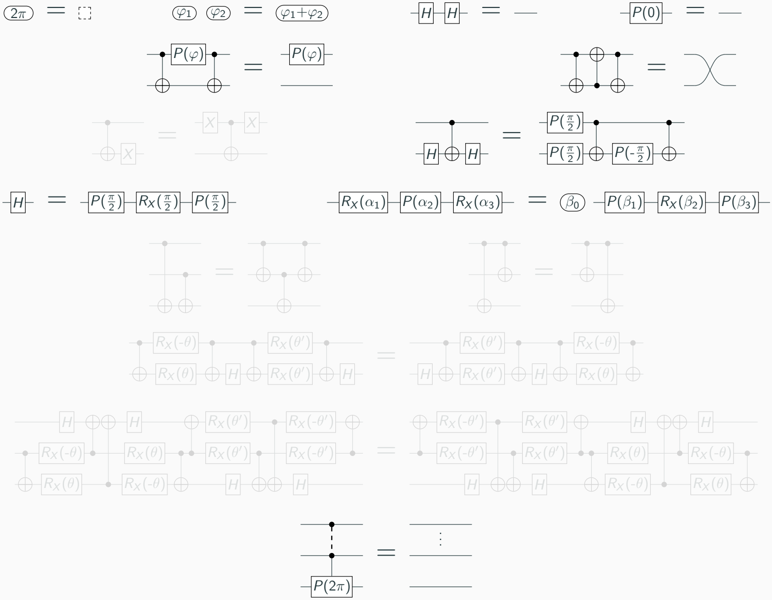

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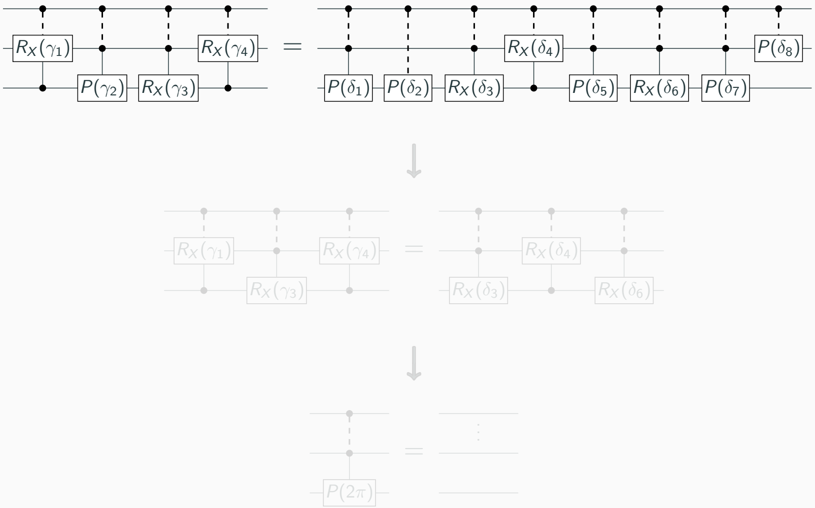

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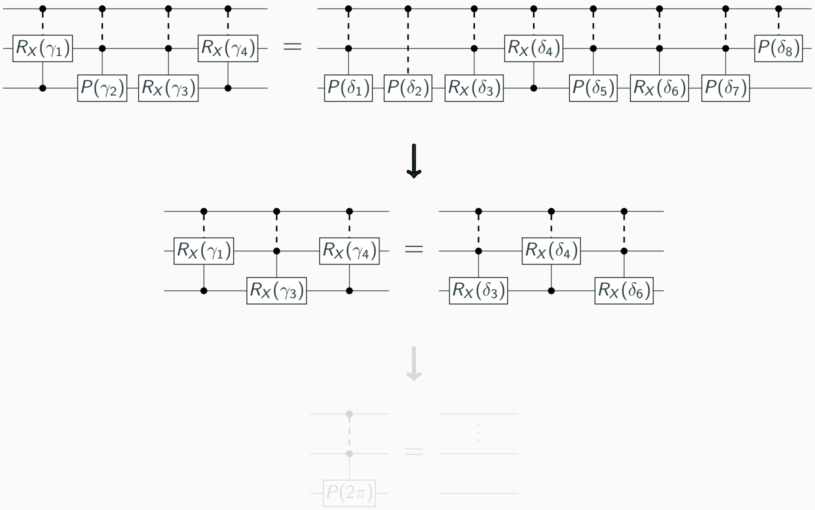

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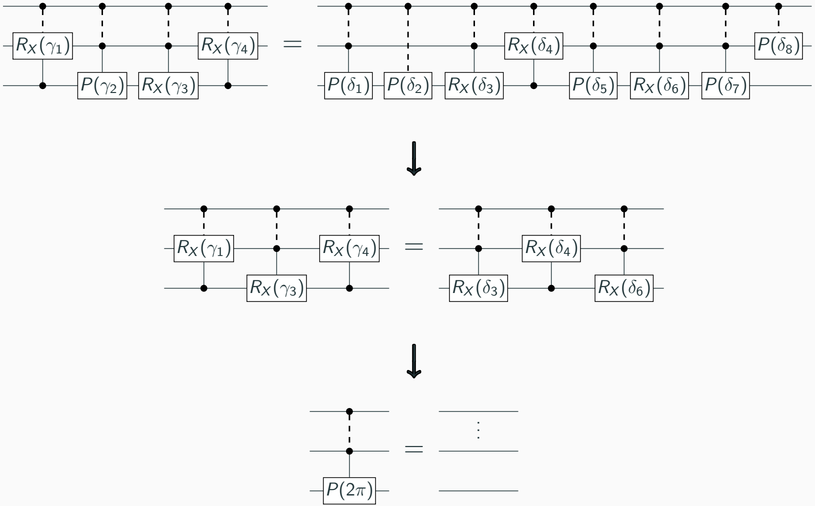

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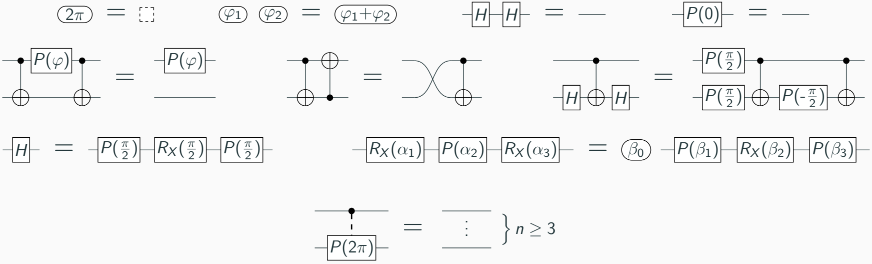

Question: Can we simplify the equational theory even more?

Theorem

This equational theory is complete, sound and minimal.

Minimality

All equations are independents.

$\forall (C_1 = C_2) \in \Gamma \quad : \quad \Gamma \backslash \{C_1 = C_2\} \nvdash C_1 = C_2$

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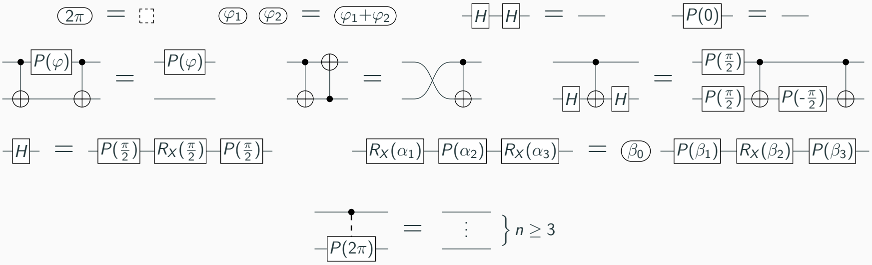

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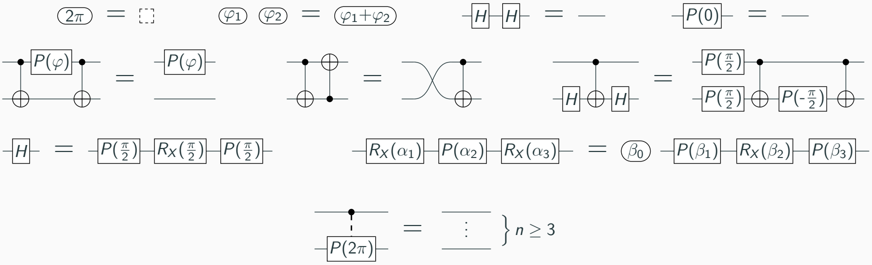

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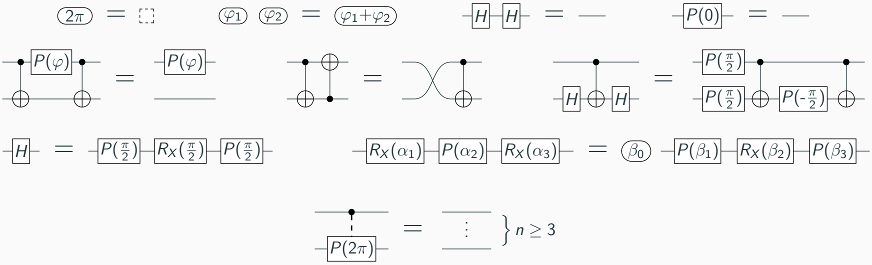

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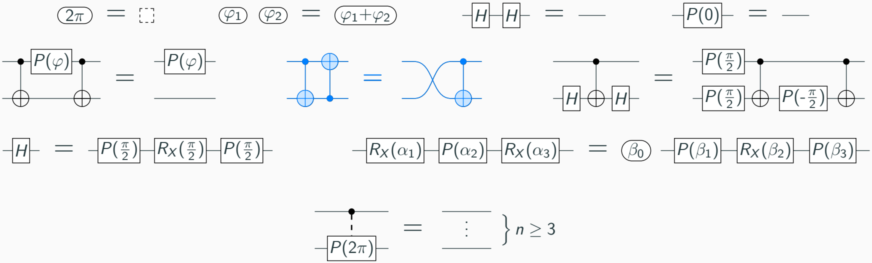
Necessity of the simple equations



For instance, the blue equation is the only one that does not preserve the parity of the number of swap gates.

Necessity of the Euler decomposition equation

Equation (E) represent a family of equations and is the only equation involving non-linear computations.

$$-\boxed{R_X(\alpha_1)}-\boxed{P(\alpha_2)}-\boxed{R_X(\alpha_3)}- \overset{(E)}{=} \ (\beta_0) \ -\boxed{P(\beta_1)}-\boxed{R_X(\beta_2)}-\boxed{P(\beta_3)}-$$

Maybe (E) is in the equational theory only to retrieve simple equations such that

$$-\boxed{P(\varphi_1)}-\boxed{P(\varphi_2)}- \overset{(P_+)}{=} -\boxed{P(\varphi_1+\varphi_2)}- \qquad -\boxed{X}-\boxed{P(\varphi)}-\boxed{X}- \overset{(P_-)}{=} \ (\varphi) \ -\boxed{P(-\varphi)}-$$

Proposition

Let Γ be a set of equations containing

- all the equations of the equational theory except (E),
- any set of instance of (E) of cardinality strictly less than $2^{\aleph_0}$,
- all instances of (P_+) and (P_-).

Then there exists an instance of (E) which is not a consequence of Γ. Hence, uncountably many instances of (E) are requiered.

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Unboundedness of the equational theory

Every instances of  are necessary (for every $n \geq 3$).

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There is no complete equational theory for quantum circuits made of equations acting on a bounded number of wires.

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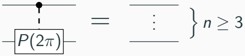
Every instances of 

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Proof sketch [1/2]

Alternative interpretation

For any $k \in \mathbb{N}$, for any quantum circuit C, let $[\![C]\!]_k^\sharp \in [0, 2\pi)$ be inductively defined as

$$[\![C_2 \circ C_1]\!]_k^\sharp = [\![C_1 \otimes C_2]\!]_k^\sharp = [\![C_2]\!]_k^\sharp + [\![C_1]\!]_k^\sharp \bmod 2\pi$$

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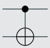
$[\![$  $]\!]^\sharp_k = [\![$  $]\!]^\sharp_k = 2^{k-2}\pi \mod 2\pi$ $\qquad [\![-\boxed{P(\varphi)}-]\!]^\sharp_k = 2^{k-1}\varphi \mod 2\pi$

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Lemme

For any n-qubit quantum circuits C_1, C_2 and $k \geq n$,

$$[\![C_1]\!] = [\![C_2]\!] \quad\Longrightarrow\quad [\![C_1]\!]^\sharp_k = [\![C_2]\!]^\sharp_k$$

Thus, any sound equation involving circuits acting on at most $n-1$ wires is also sound according to $[\![\cdot]\!]^\sharp_{n-1}$.

However,

$[\![$ $\}n\ ]\!]^\sharp_{n-1} = \pi \neq 0 = [\![$ $\}n\ ]\!]^\sharp_{n-1}$

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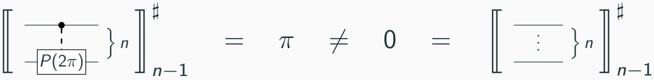

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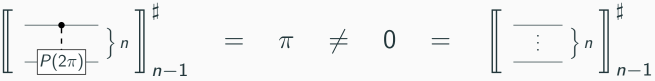
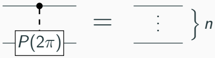


Hence 

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Discussion of the theorem

Possible weakness: $[\![C]\!]_k^\sharp$ is closely related to the determinant of $[\![C]\!]$. What if we consider quantum circuits up to global phases?

$\longrightarrow$ The theorem still holds!

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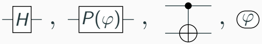
Possible weakness: The choice of the generators 

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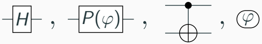

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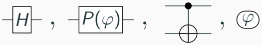
Possible weakness: The choice of the generators 

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Corollary

Any complete equational theory for the fragment where parameters are multiple of $\frac{\pi}{2^n}$ must contain at least one equation acting on $n+2$ wires.

For Clifford quantum circuits (case $n=1$),
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For Clifford+T quantum circuits (case $n=2$),
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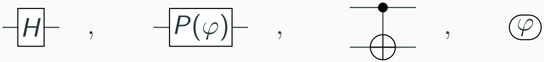
Extension to quantum circuits with ancillae

Quantum circuits with ancillae are generated by



together with

$$\vdash \quad \text{and} \quad \dashv$$

respectively denoting qubit initialisation and qubit termination.

(The generator $\dashv$ can only be applied to qubits in the $|0\rangle$-state.)

Semantics

We extend $[\![\cdot]\!]$ with $[\![\vdash]\!] = |0\rangle$ and $[\![\dashv]\!] = \langle 0|$.

Universal for isometries

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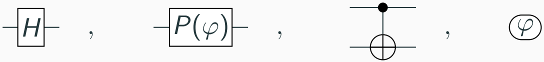

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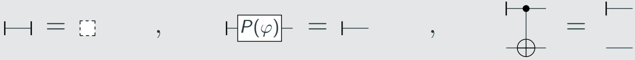
Boundedness of the equational theory with ancillae

Theorem [Clément,Delorme,Perdrix,Vilmart'2024]

Adding those three equations makes the equational theory complete for quantum circuits with ancillae.



Using ancillae, we can build controlled gates without dividing the angles.

In these more general settings,

is derivable for $n \geq 4$.

Hence, using ancillae, there is a complete equational theory made of equations acting on at most 3 wires.

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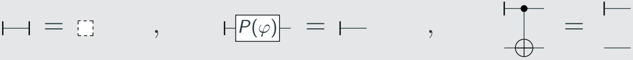

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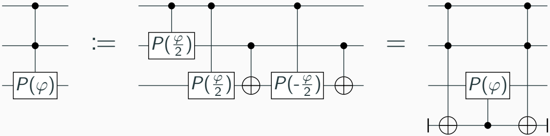

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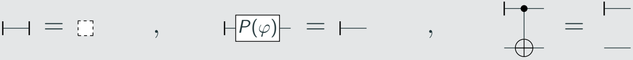

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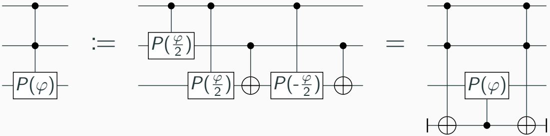
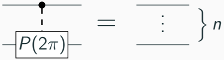


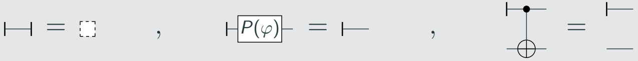
In these more general settings,  is derivable for $n \geq 4$.

Hence, using ancillae, there is a complete equational theory made of equations acting on at most 3 wires.

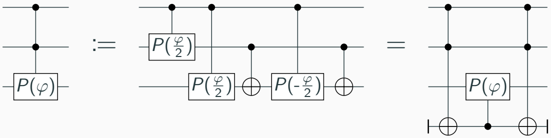
Boundedness of the equational theory with ancillae

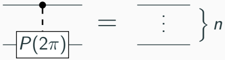
Theorem [Clément,Delorme,Perdrix,Vilmart'2024]

Adding those three equations makes the equational theory complete for quantum circuits with ancillae.



Using ancillae, we can build controlled gates without dividing the angles.



In these more general settings, 

is derivable for $n \geq 4$.

Hence, using ancillae, there is a complete equational theory made of equations acting on at most 3 wires.

Thanks

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Minimal Equational Theories for Quantum Circuits

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