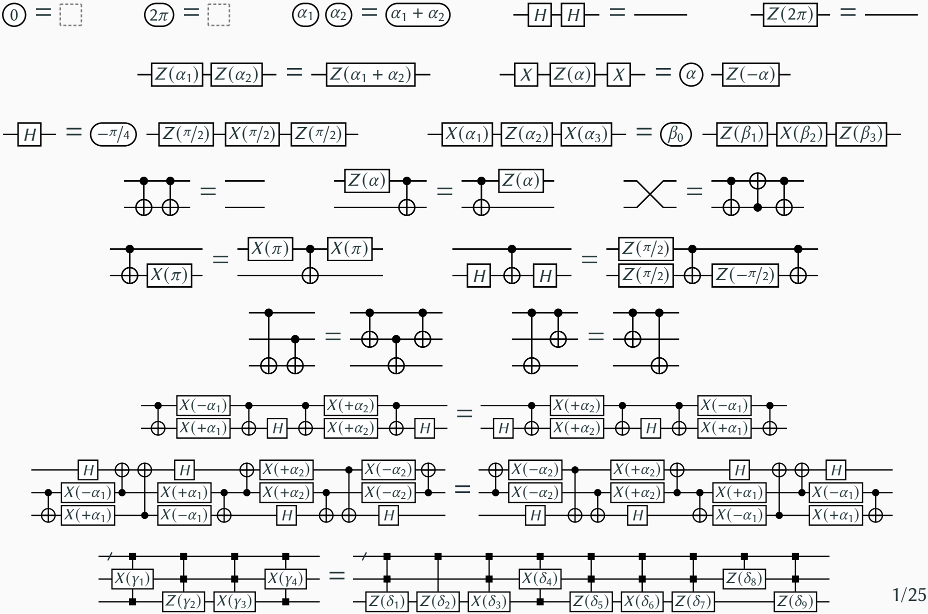
We Improved It, Proved It Minimal, Then Cheated.

9th April 2026 - MOCQUA Seminar

Noé Delorme

Université de Lorraine, CNRS, INRIA, LORIA, Nancy, France

Before



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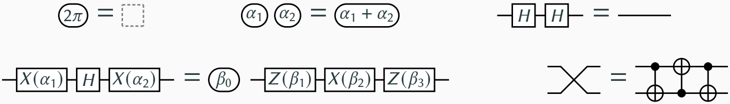
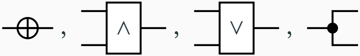


What are quantum circuits?

Boolean circuits

are made of simple generators



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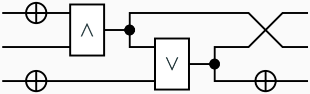

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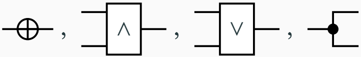

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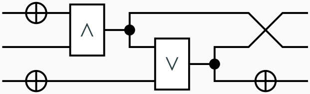

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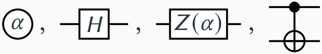

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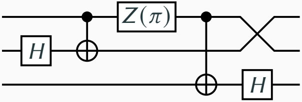


Other gates as shorthand notations

Other usual gates can be defined as shorthand notations by compositions of the generators.

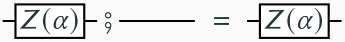
$$-\boxed{X(\alpha)}- \;:=\; -\boxed{H}-\boxed{Z(\alpha)}-\boxed{H}- \qquad -\boxed{Y(\alpha)}- \;:=\; -\boxed{Z(-\pi/2)}-\boxed{X(\alpha)}-\boxed{Z(+\pi/2)}-$$

$$-\boxed{X}- \;:=\; -\boxed{X(\pi)}- \qquad -\boxed{Y}- \;:=\; -\boxed{Y(\pi)}- \qquad -\boxed{Z}- \;:=\; -\boxed{Z(\pi)}-$$

$$-\boxed{S}- \;:=\; -\boxed{Z(\pi/2)}- \qquad -\boxed{T}- \;:=\; -\boxed{Z(\pi/4)}-$$

PROP formalism

Formally, quantum circuits are defined in the PROP formalism, which brings coherence equations such that



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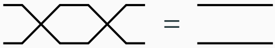

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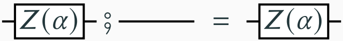

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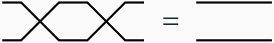

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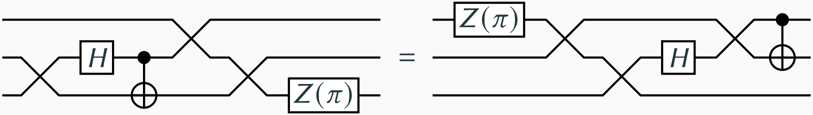

Interpretation of quantum circuits

Interpretation

$$\llbracket C_1 \mathbin{;} C_2 \rrbracket = \llbracket C_2 \rrbracket \circ \llbracket C_1 \rrbracket \qquad \llbracket C_1 \otimes C_2 \rrbracket = \llbracket C_1 \rrbracket \otimes \llbracket C_2 \rrbracket$$

$\llbracket$  $\rrbracket = (1)$ $\qquad \llbracket \textcircled{\alpha} \rrbracket = (e^{i\alpha})$

$$\llbracket \text{———} \rrbracket = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \llbracket -\boxed{H}- \rrbracket = {}^1\!/\!\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \llbracket -\boxed{Z(\alpha)}- \rrbracket = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

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circuits $\neq$ matrices

Interpretation

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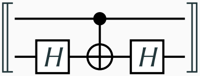
Distinct circuits can have the same interpretation.



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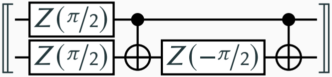

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Which circuit is the best? In terms of total number of gates ; in terms of number of fault-tolerant gates ; or to satisfy hardware constraints (for instance topological constraints).

Motivations

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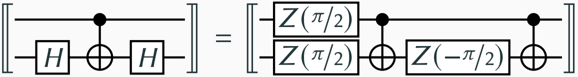

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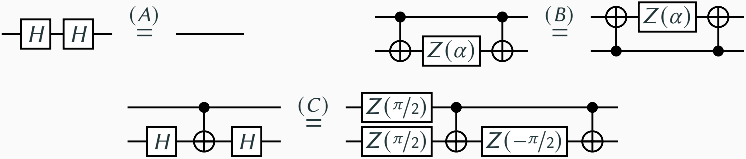

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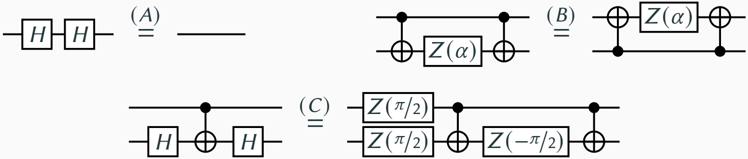

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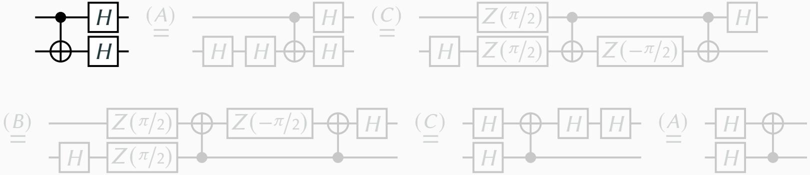

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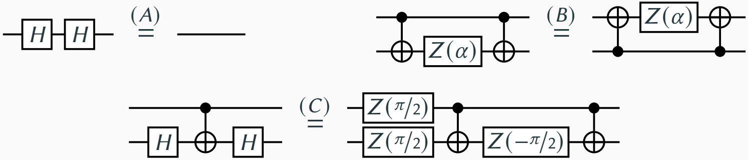

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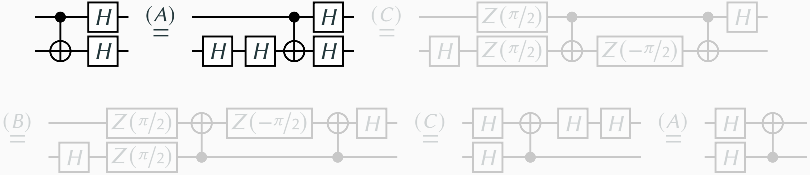

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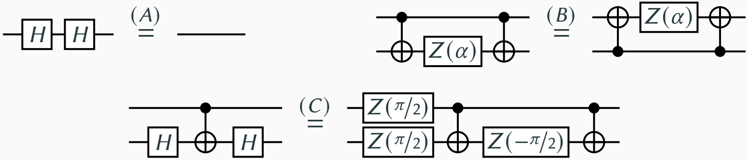

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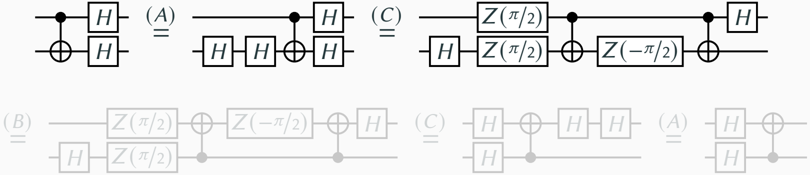

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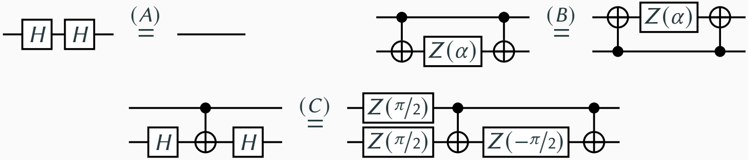

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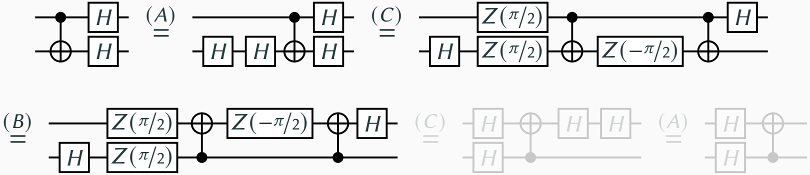

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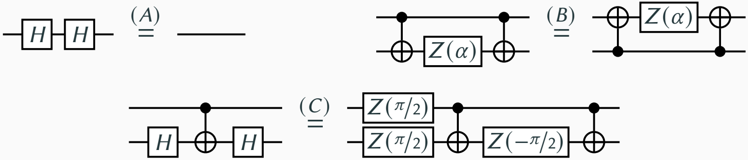

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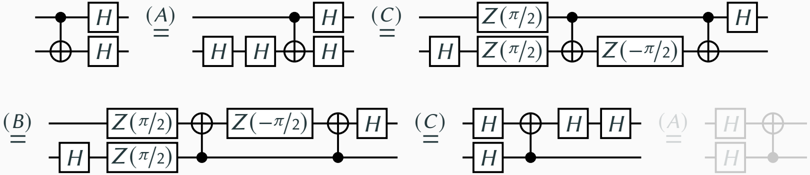

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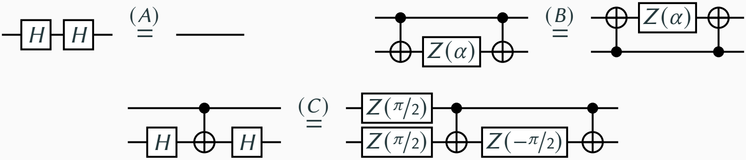

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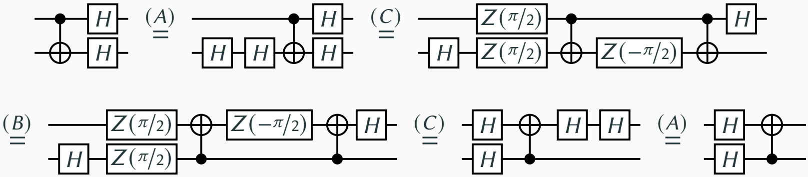

Soundness and completeness

Is there a set of axioms $\mathcal{R}$ from which we can derive any true equation and only true equations in a graphical language $\mathbf{P}$?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \quad : \quad \mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket$$

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[Clément,Heurtel,Mansfield,Perdrix,Valiron'2023]

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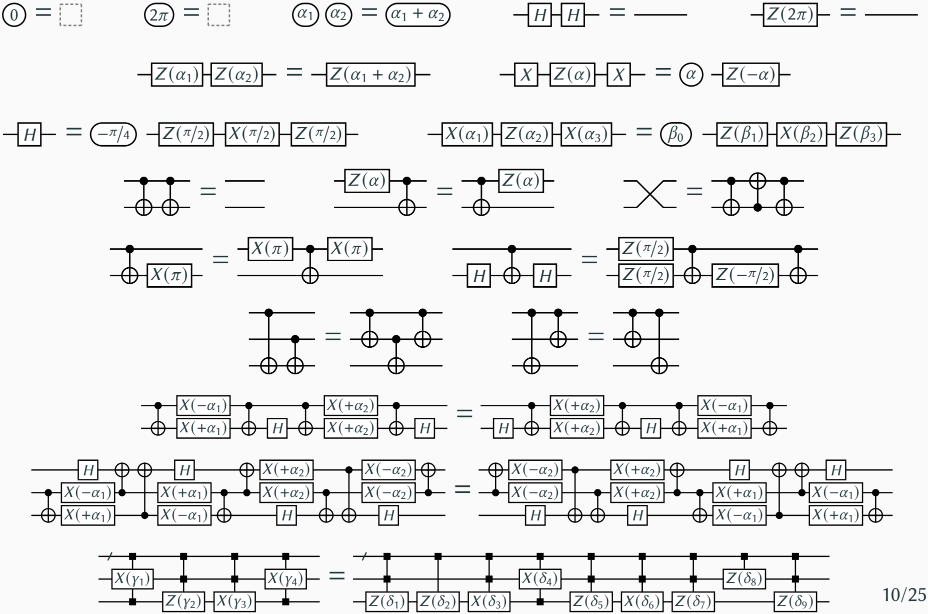

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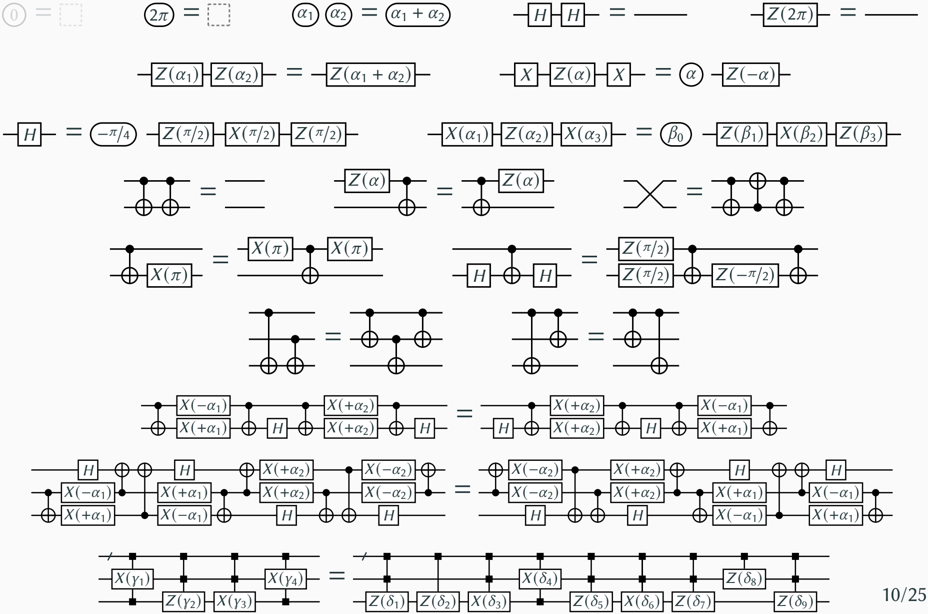

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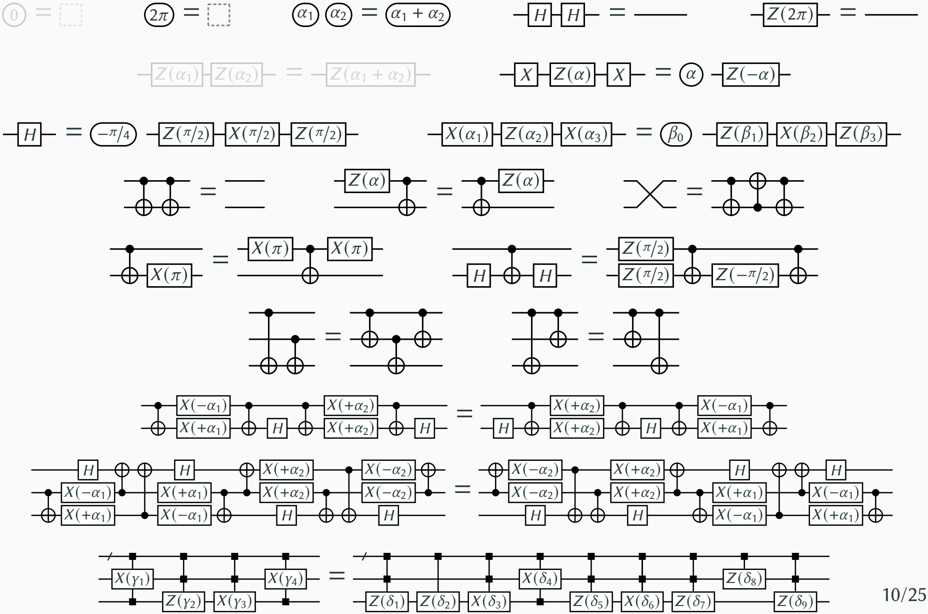

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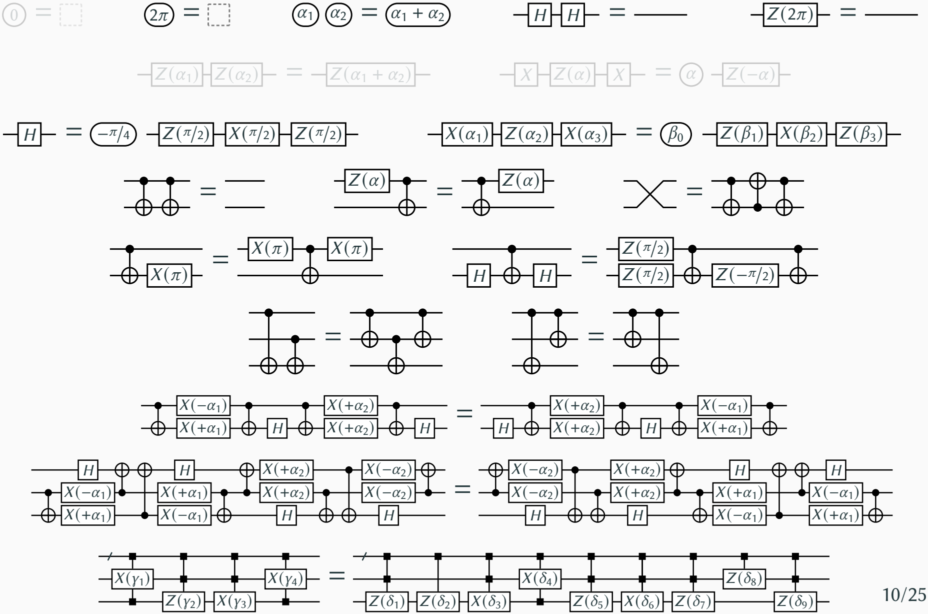

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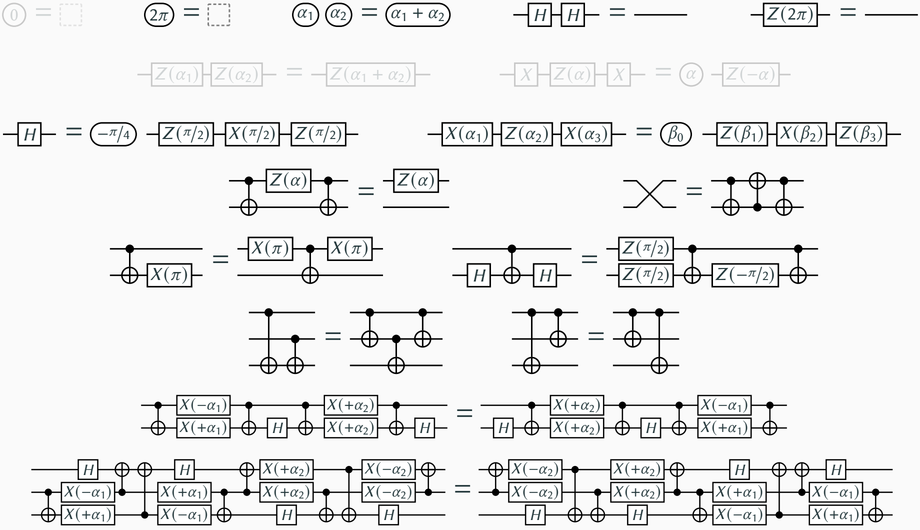

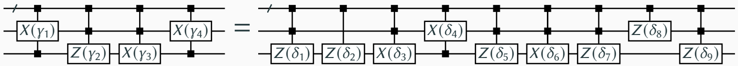

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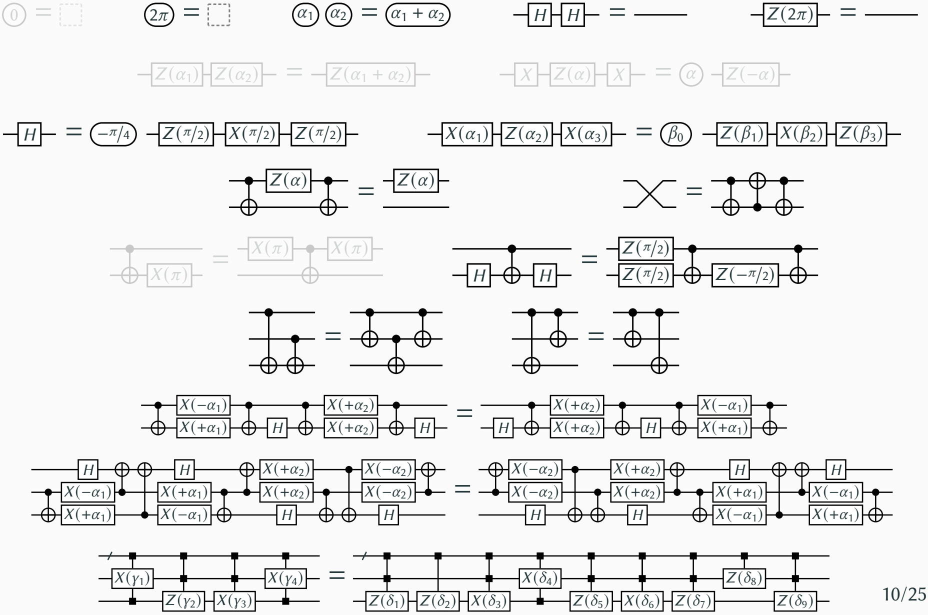

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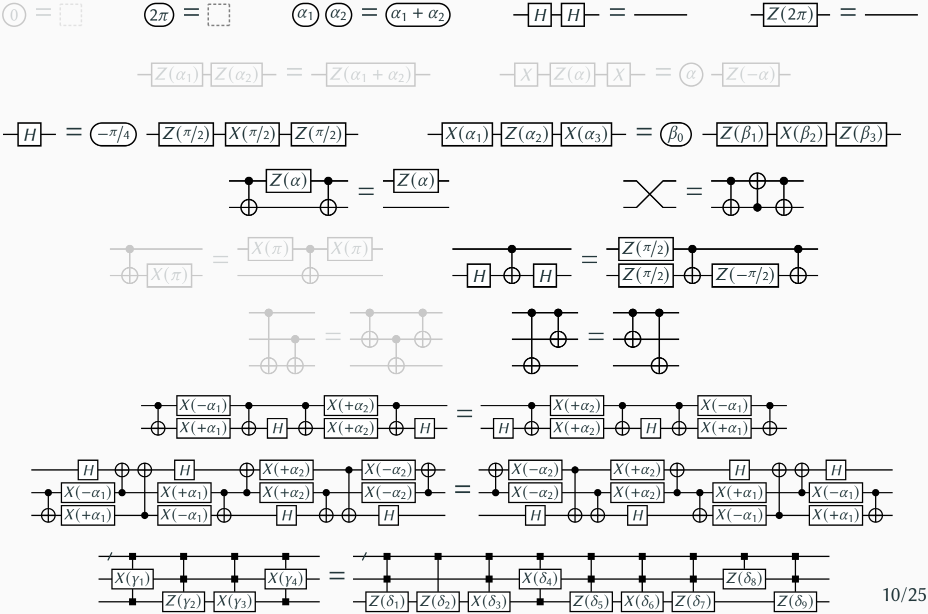

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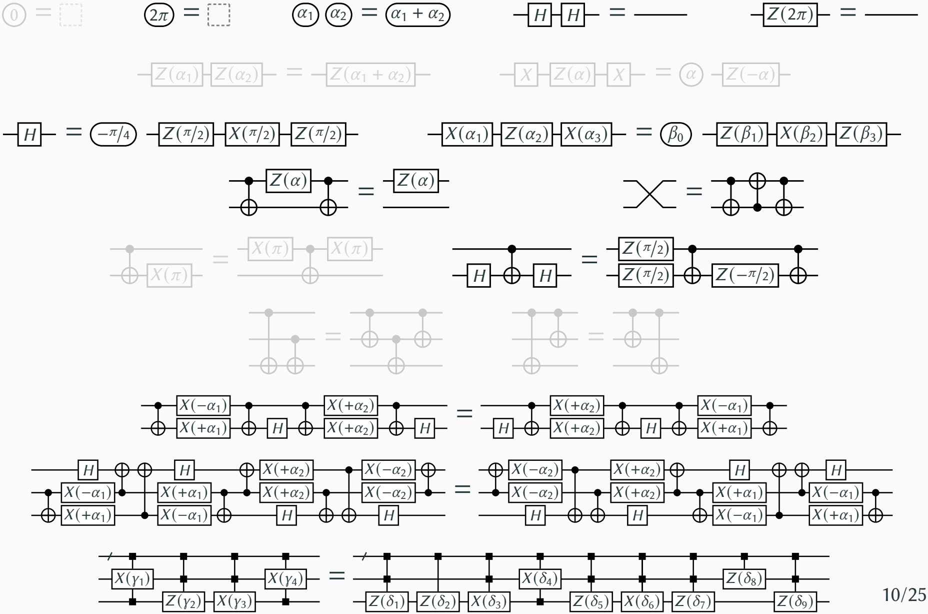

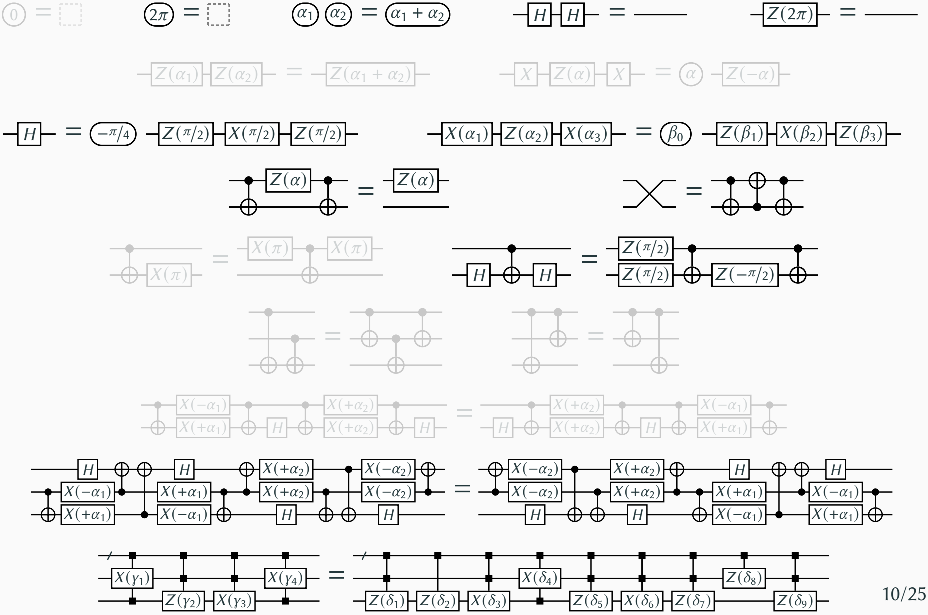
0 =
2π =
α1 α2 = α1 + α2
H H =
Z(2π) =
Z(α1) Z(α2) = Z(α1 + α2)
X Z(α) X = α Z(−α)
H = −π/4 Z(π/2) X(π/2) Z(π/2)
X(α1) Z(α2) X(α3) = β0 Z(β1) X(β2) Z(β3)
Z(α) = Z(α)
=
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=
=
X(−α1) X(+α1) H X(+α2) X(+α2) H = H X(+α2) X(+α2) H X(−α1) X(+α1)
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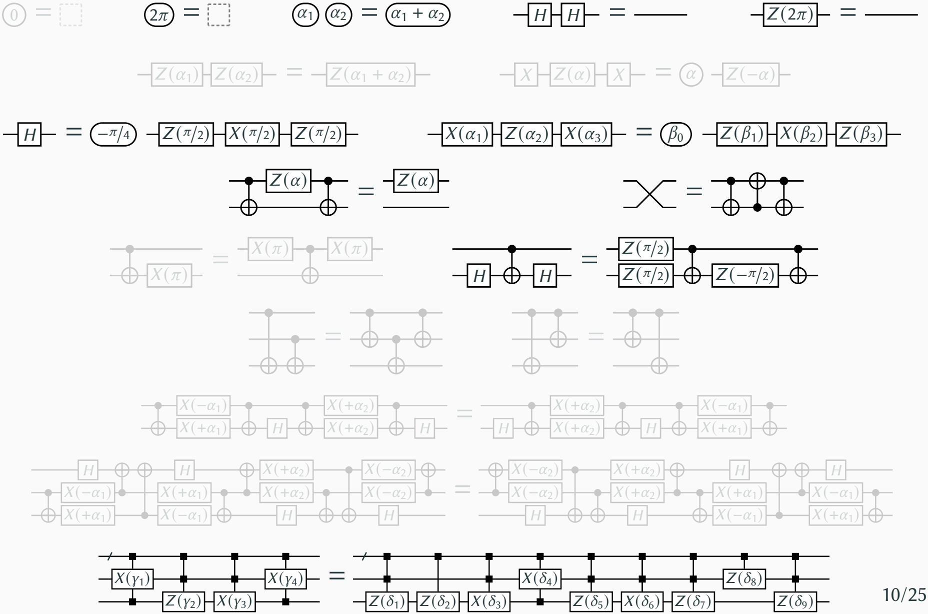
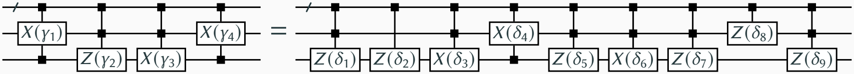
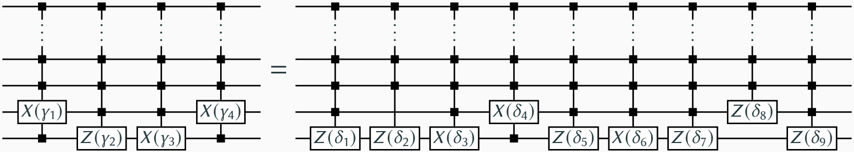




There is an instance of this axiom for any number of wires $n \geq 2$, and all gates are shorthand notations implementing controlled operations. For instance, the controlled version of $\boxed{Z(\alpha)}$ is defined as follows.

The huge and ugly axiom



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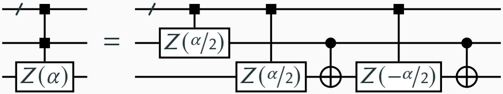
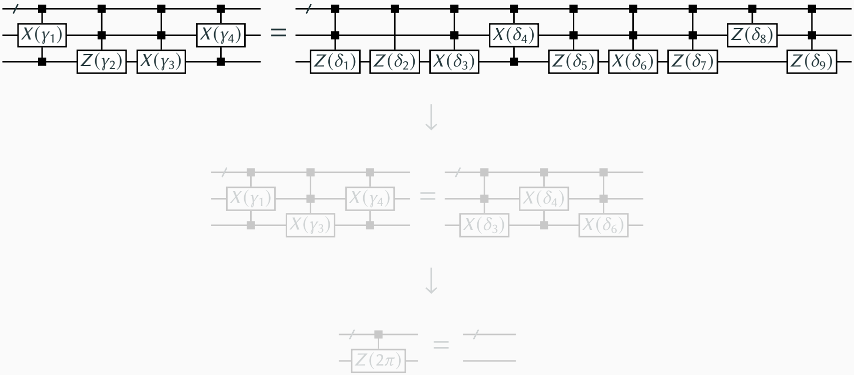


Huge, ugly, and useless...



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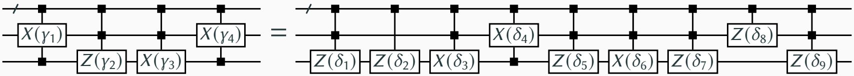
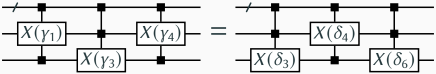


$\downarrow$



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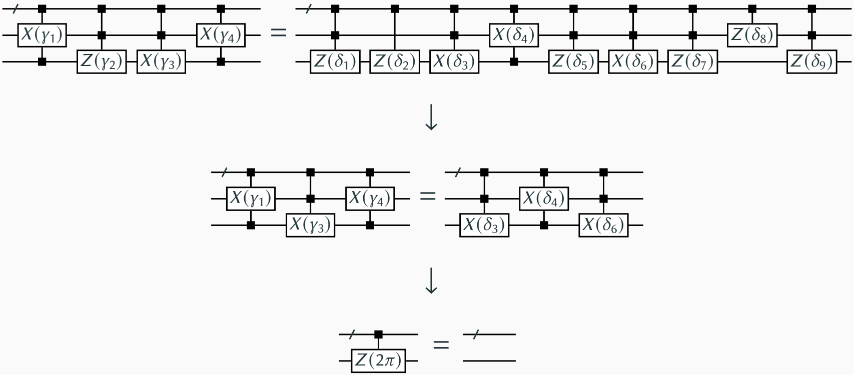
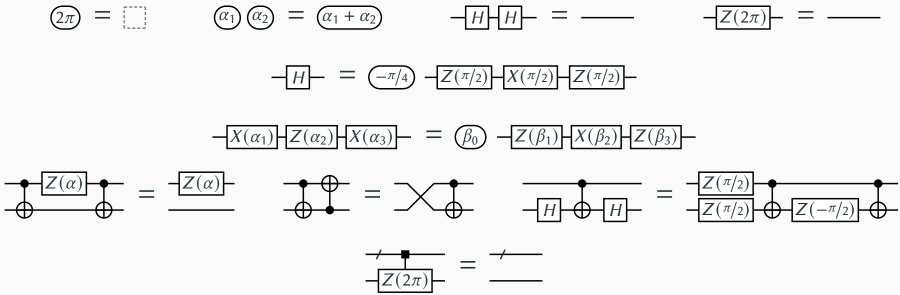


New set of axioms



Theorem

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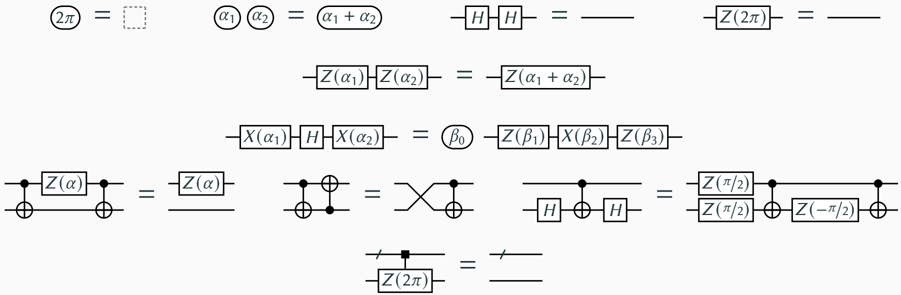

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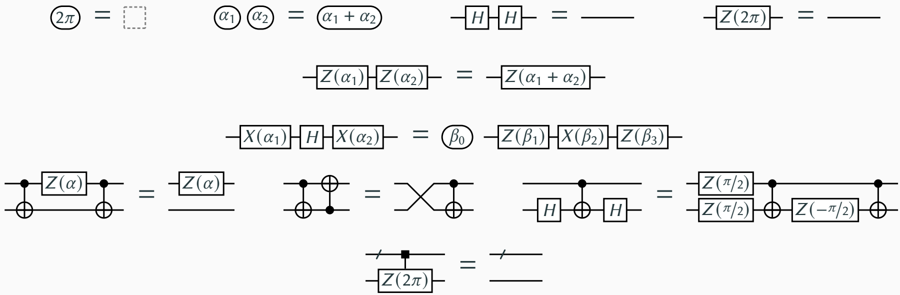

Theorem

The axioms are independents.

Really?

- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
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- Let's cheat!

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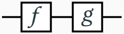
“If the game is impossible, then change the rules.”

Generators

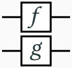
id_0 $\qquad$ id_1 $\qquad$ σ $\qquad$ a $\qquad$ b

Composition operators

$$\frac{f : n \to p \quad g : p \to m}{f \fatsemi g : n \to m}$$



$$\frac{f : n \to m \quad g : p \to q}{f \otimes g : n + p \to m + q}$$

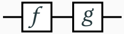


NEW!

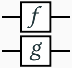
$$\frac{f : n \to n}{\mathrm{C}(f) : 1 + n \to 1 + n}$$

Composition operators

$$\frac{f : n \longrightarrow p \qquad g : p \longrightarrow m}{f \fatsemi g : n \longrightarrow m}$$



$$\frac{f : n \longrightarrow m \qquad g : p \longrightarrow q}{f \otimes g : n + p \longrightarrow m + q}$$



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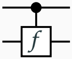
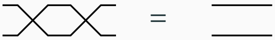
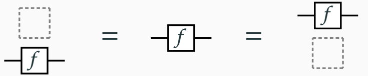


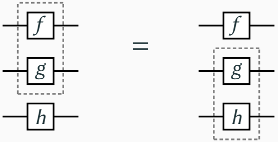
$$\sigma \fatsemi \sigma \quad = \quad \mathrm{id}_1 \otimes \mathrm{id}_1$$



$$\mathrm{id}_0 \otimes f \quad = \quad f \quad = \quad f \otimes \mathrm{id}_0$$



$$(f \otimes g) \otimes h \quad = \quad f \otimes (g \otimes h)$$

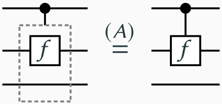


$$(f \fatsemi g) \fatsemi h \quad = \quad f \fatsemi (g \fatsemi g)$$

$$(f_1 \fatsemi g_2) \otimes (f_2 \fatsemi g_2) \quad = \quad (f_1 \otimes f_2) \fatsemi (g_1 \otimes g_2)$$

. . .

NEW!

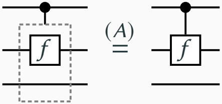


NEW!

NEW!

Coherence equations

NEW!



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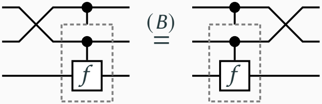

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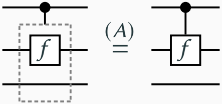

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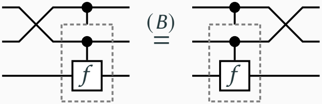

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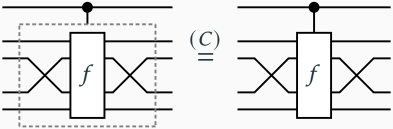

Controlled PROPs

Definition

A control functor maps any endocircuit $f : n \to n$ to the endocircuit $C(f) : (1+n) \to (1+n)$ while satisfying (A), (B), and (C).

Definition

A controlled PROP is a PROP equipped with a control functor.

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A controlled PROP is a PROP equipped with a control functor.

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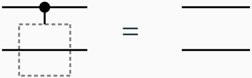
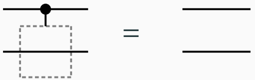
$$\mathrm{C}(\mathrm{id}_1) = \mathrm{id}_1 \otimes \mathrm{id}_1$$

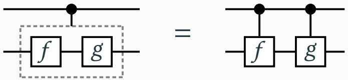


$$\mathrm{C}(f \mathbin{;} g) = \mathrm{C}(f) \mathbin{;} \mathrm{C}(g)$$

Notice however that $\mathrm{C}(f \otimes g) \neq \mathrm{C}(f) \otimes \mathrm{C}(g)$, instead we have

$$C(\mathrm{id}_1) = \mathrm{id}_1 \otimes \mathrm{id}_1 \qquad C(f \fatsemi g) = C(f) \fatsemi C(g)$$





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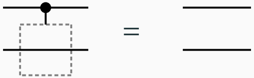

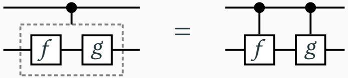

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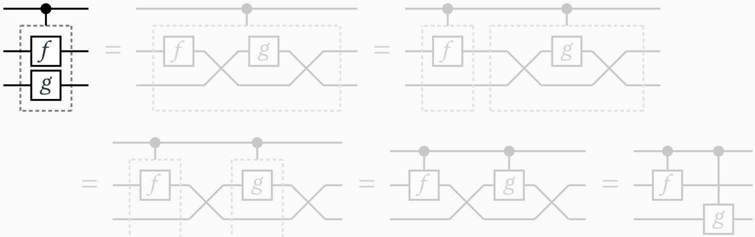

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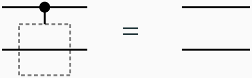

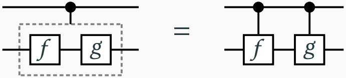

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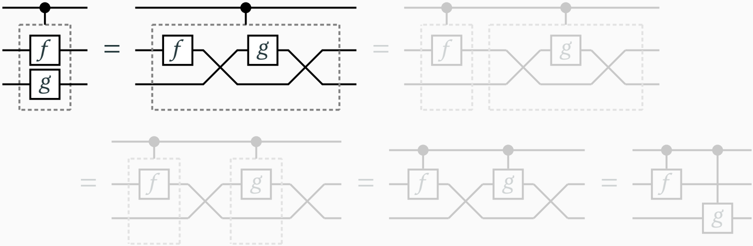

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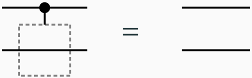

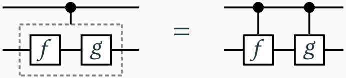

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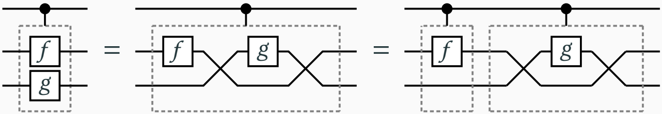

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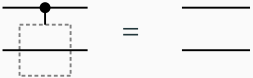

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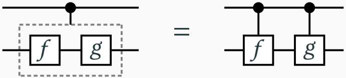

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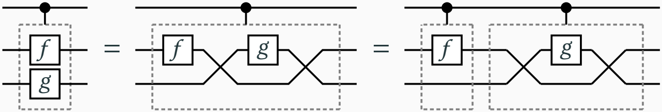

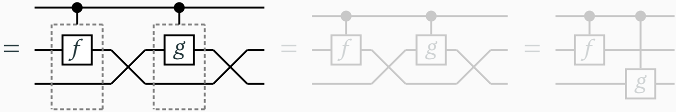

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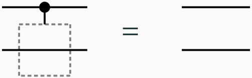

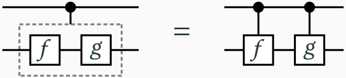

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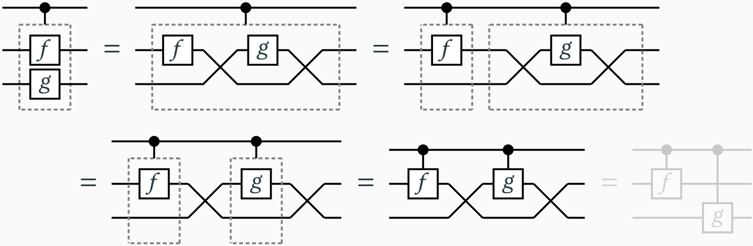

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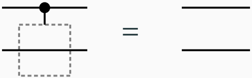

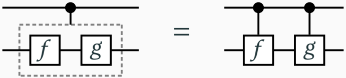

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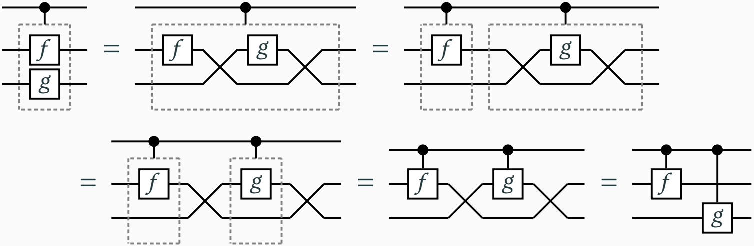

$$\mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \mathbf{P}/\mathcal{R} \vdash \mathrm{C}(C_1) = \mathrm{C}(C_2) \quad \textbf{NEW!}$$

Example

Let **CNot** be the controlled PROP generated by with the following axioms.

Then, is not a generator. It is the circuit C( ).

Moreover, the equations is = is derivable.

Does anyone actually read the headlines anyway?

$$\mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \mathbf{P}/\mathcal{R} \vdash \mathrm{C}(C_1) = \mathrm{C}(C_2) \quad \textbf{NEW!}$$

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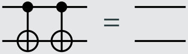
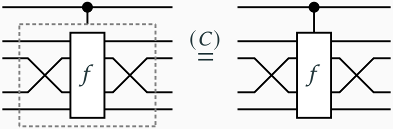
Moreover, the equations is  is derivable.

Conjugated PROP



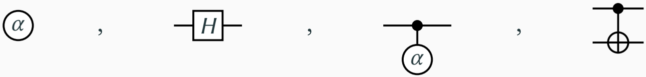
Definition

A conjugated PROP is a controlled PROP such that

is satisfied whenever $\mathbf{P}/\mathcal{R} \vdash$  $=$.

Simpler generators

Simpler generators



Simpler generators

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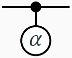

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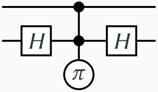
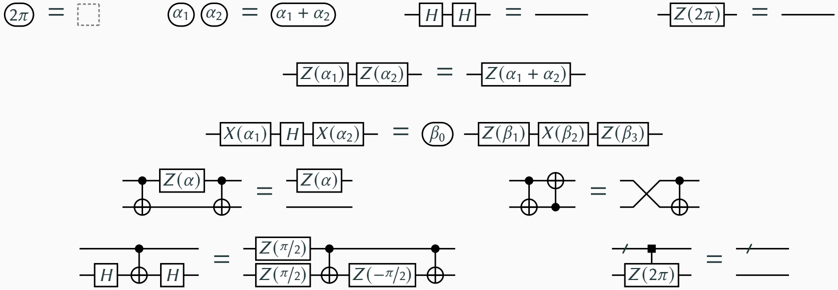
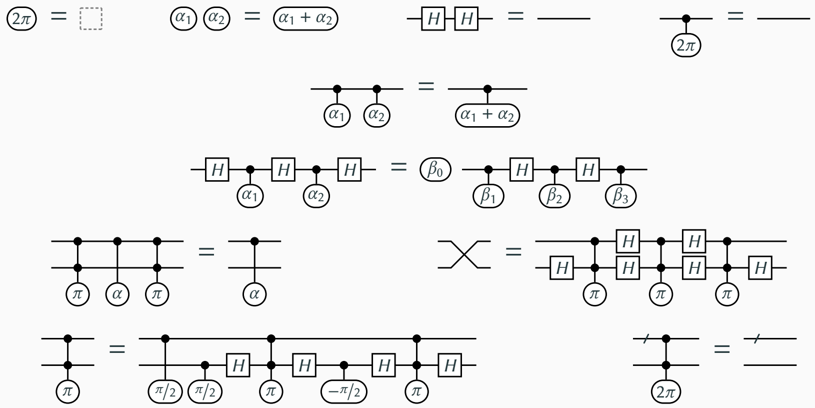


Simpler generators

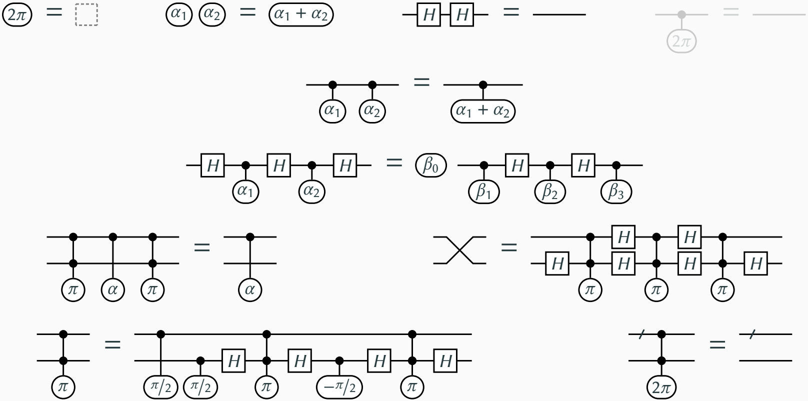
Simpler axioms

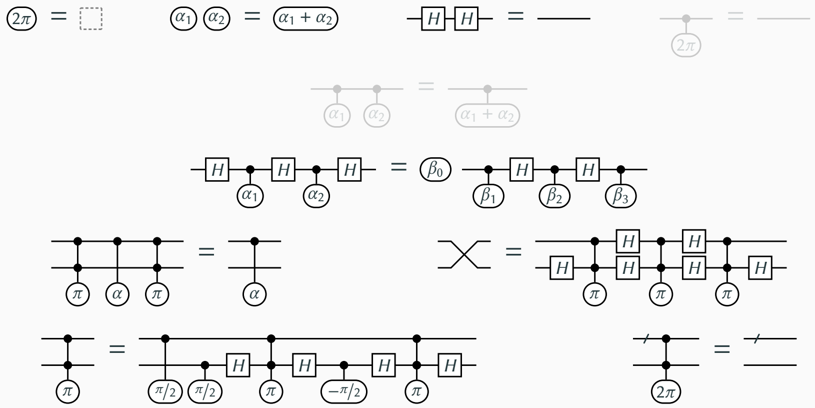


Simpler axioms

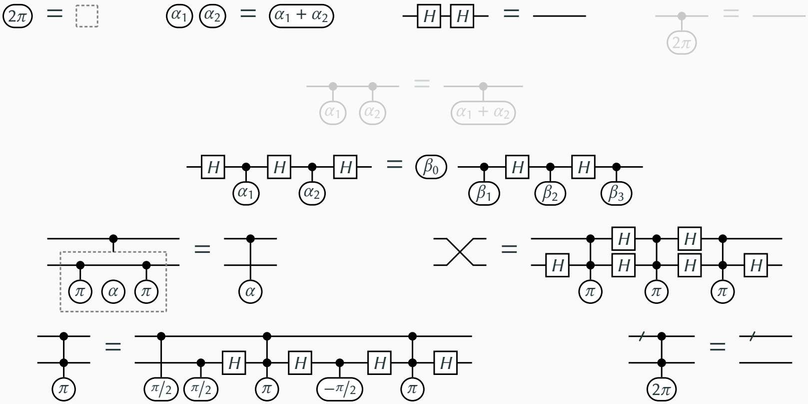


Simpler axioms




2π
α1 α2
α1 + α2
H H
2π
α1 α2
α1 + α2
H H H
β0
H H
α1 α2
β1 β2 β3
π α π
α
H H H H H
π π π
H H H H
π/2 π/2 π −π/2 π
π
2π

Simpler axioms



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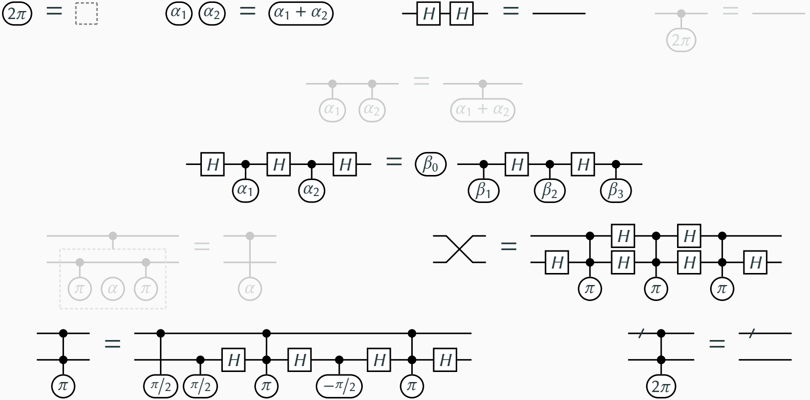

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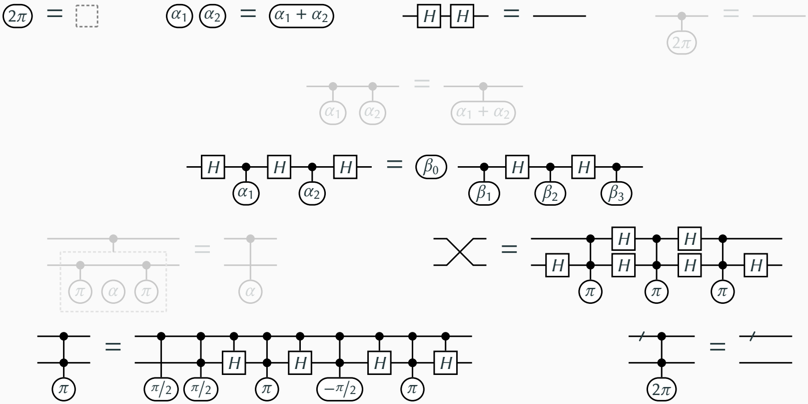

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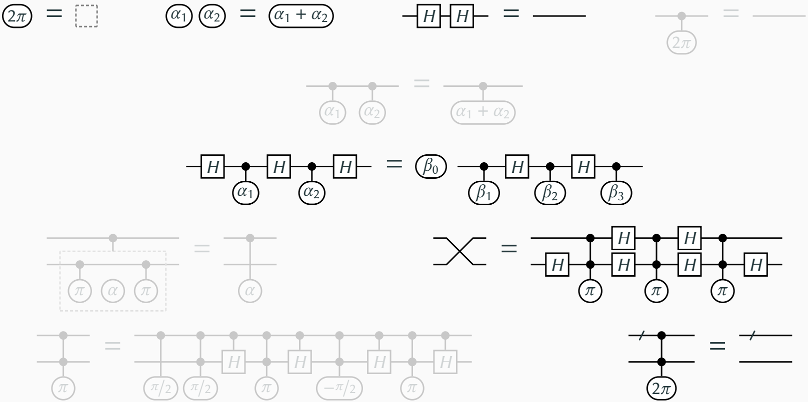

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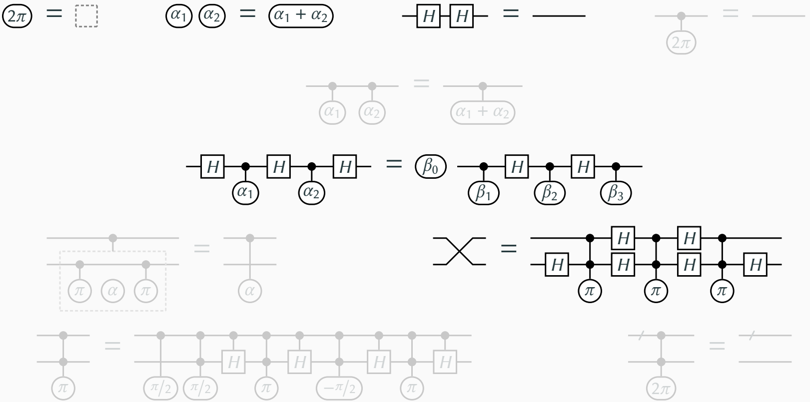
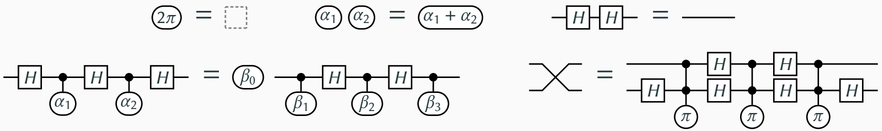
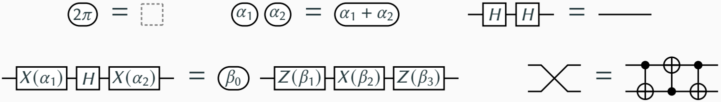


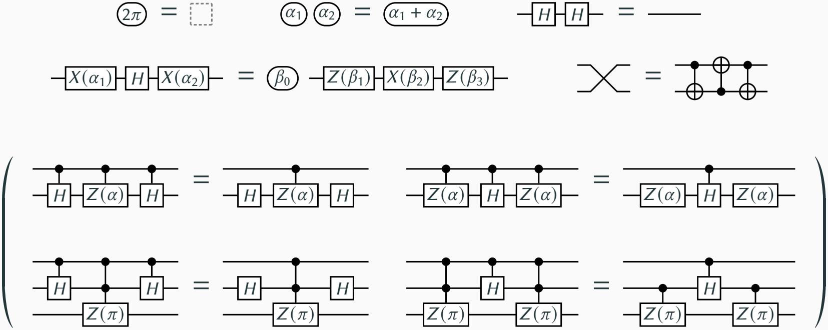
Final set of axioms using controlled PROPs



Final set of axioms using controlled PROPs



Final set of axioms using controlled PROPs



Thanks

(arXiv:2508.21756)

Diagrammatic Reasoning with Control as a Constructor, Applications to Quantum Circuits