

We Improved It, Proved It Minimal, Then Cheated.

9th April 2026 - MOCQUA Seminar

Noé Delorme

Université de Lorraine, CNRS, INRIA, LORIA, Nancy, France

Before

$$\textcircled{2\pi} = \square$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$- \boxed{Z(\alpha_1)} - \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$- [H] - = (-\pi/4) - [Z(\pi/2)] - [X(\pi/2)] - [Z(\pi/2)] -$$

The diagram shows a circuit with two horizontal lines. The top line has two black dots (control qubits) and the bottom line has two white circles with black outlines (target qubits). Vertical lines connect each control dot to its corresponding target circle, representing two CNOT gates. This is followed by an equals sign and a simplified circuit with two horizontal lines and no gates, indicating that the two CNOT gates cancel each other out.

A diagram showing a crossing of two lines (left) is equal to a 2x2 grid of nodes (right). The nodes are arranged in two rows and two columns. The top row has a solid black dot, a circle with a plus sign, and a solid black dot. The bottom row has a circle with a plus sign, a solid black dot, and a circle with a plus sign. Vertical lines connect the nodes in each column.

The diagram shows two equivalent quantum circuit representations. On the left, a CNOT gate (control on the top line, target on the bottom line) is followed by a phase gate $X(\pi)$ on the target line. This is shown to be equivalent to a phase gate $X(\pi)$ on the control line followed by a CNOT gate.

1/25

After

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\boxed{H} \boxed{H} = \text{---}$$

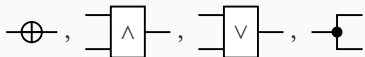
$$\boxed{X(\alpha_1)} \boxed{H} \boxed{X(\alpha_2)} = \textcircled{\beta_0} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)}$$

$$\text{X} = \begin{array}{c} \bullet \oplus \bullet \\ | \quad | \\ \oplus \bullet \oplus \end{array}$$

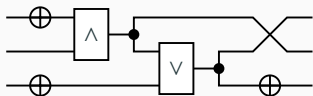
What are quantum circuits?

Boolean circuits

are made of simple generators

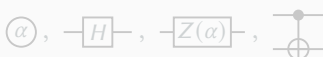


and can be composed to form circuits expressing classical algorithms.



Quantum circuits

are made of simple generators



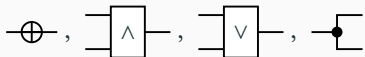
and can be composed to form circuits expressing quantum algorithms.



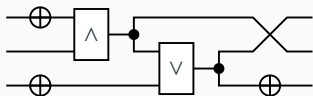
What are quantum circuits?

Boolean circuits

are made of simple generators

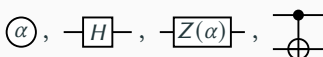


and can be composed to form circuits expressing classical algorithms.

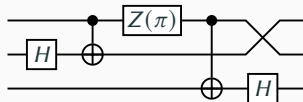


Quantum circuits

are made of simple generators



and can be composed to form circuits expressing quantum algorithms.



Other gates as shorthand notations

Other usual gates can be defined as **shorthand notations** by compositions of the generators.

$$\text{---} \boxed{X(\alpha)} \text{---} := \text{---} \boxed{H} \boxed{Z(\alpha)} \boxed{H} \text{---} \qquad \text{---} \boxed{Y(\alpha)} \text{---} := \text{---} \boxed{Z(-\pi/2)} \boxed{X(\alpha)} \boxed{Z(+\pi/2)} \text{---}$$

$$\text{---} \boxed{X} \text{---} := \text{---} \boxed{X(\pi)} \text{---} \qquad \text{---} \boxed{Y} \text{---} := \text{---} \boxed{Y(\pi)} \text{---} \qquad \text{---} \boxed{Z} \text{---} := \text{---} \boxed{Z(\pi)} \text{---}$$

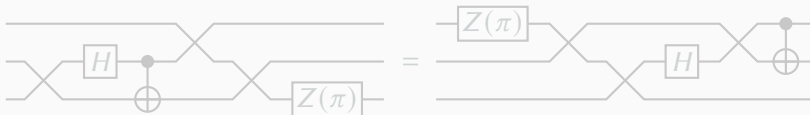
$$\text{---} \boxed{S} \text{---} := \text{---} \boxed{Z(\pi/2)} \text{---} \qquad \text{---} \boxed{T} \text{---} := \text{---} \boxed{Z(\pi/4)} \text{---}$$

PROP formalism

Formally, quantum circuits are defined in the **PROP** formalism, which brings coherence equations such that

$$\boxed{Z(\alpha)} \circ \text{---} = \boxed{Z(\alpha)} \quad \text{and} \quad \text{X-crossing} = \text{---}$$

to captures the intuitive behaviour of wires and boxes. Then quantum circuits are defined “up to deformation”.

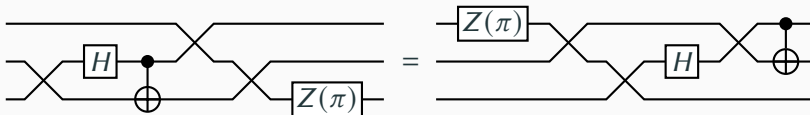


PROP formalism

Formally, quantum circuits are defined in the **PROP** formalism, which brings coherence equations such that

$$\boxed{Z(\alpha)} \circ \text{---} = \boxed{Z(\alpha)} \quad \text{and} \quad \text{---} \times \text{---} = \text{---}$$

to captures the intuitive behaviour of wires and boxes. Then quantum circuits are defined “**up to deformation**”.



Interpretation of quantum circuits

Interpretation

$$[[C_1 \circ C_2]] = [[C_2]] \circ [[C_1]] \quad [[C_1 \otimes C_2]] = [[C_1]] \otimes [[C_2]]$$

$$[[\text{---}\square\text{---}]] = (1) \quad [[\text{---}\bigcirc^{\alpha}\text{---}]] = (e^{i\alpha})$$

$$[[\text{---}]] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [[\text{---}\square{H}\text{---}]] = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad [[\text{---}\square{Z(\alpha)}\text{---}]] = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$[[\text{---}\bullet\text{---}\oplus\text{---}]] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad [[\text{---}\times\text{---}]] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

circuits $\neq$ matrices

Interpretation of quantum circuits

Interpretation

$$[[C_1 \circ C_2]] = [[C_2]] \circ [[C_1]] \quad [[C_1 \otimes C_2]] = [[C_1]] \otimes [[C_2]]$$

$$[[\text{---}\square\text{---}]] = (1) \quad [[\text{---}\bigcirc\alpha\text{---}]] = (e^{i\alpha})$$

$$[[\text{---}]] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [[\text{---}\square H\text{---}]] = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad [[\text{---}\square Z(\alpha)\text{---}]] = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$[[\text{---}\bullet\text{---}\oplus\text{---}]] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad [[\text{---}\times\text{---}]] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

circuits $\neq$ matrices

Motivations

Distinct circuits can have the same interpretation.

$$\left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] = \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \boxed{Z(\pi/2)} \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \boxed{Z(\pi/2)} \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \boxed{Z(-\pi/2)} \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Which circuit is the best? In terms of total number of gates ; in terms of number of fault-tolerant gates ; or to satisfy hardware constraints (for instance topological constraints).

Motivations

Distinct circuits can have the same interpretation.

$$\left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] = \left[\begin{array}{c} \text{---} \boxed{Z(\pi/2)} \text{---} \\ | \\ \text{---} \boxed{Z(\pi/2)} \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \text{---} \boxed{Z(-\pi/2)} \text{---} \\ | \\ \text{---} \end{array} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Which circuit is the best? In terms of total number of gates ; in terms of number of fault-tolerant gates ; or to satisfy hardware constraints (for instance topological constraints).

Circuit rewriting

Given a set of axioms,

$$\text{---} \boxed{H} \text{---} \boxed{H} \text{---} \stackrel{(A)}{=} \text{---}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(B)}{=} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{Z(\alpha)} \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} \boxed{Z(\pi/2)} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array}$$

we can derive other equations. For instance,

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(A)}{=} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(B)}{=} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(A)}{=} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array}$$

Circuit rewriting

Given a set of axioms,

$$\text{---} [H] [H] \text{---} \stackrel{(A)}{=} \text{---}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} [Z(\alpha)] \begin{array}{c} \bullet \\ \oplus \end{array} \stackrel{(B)}{=} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} [H] \\ \oplus \end{array} [H] \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} [Z(-\pi/2)] \begin{array}{c} \bullet \\ \oplus \end{array}$$

we can derive other equations. For instance,

$$\begin{array}{c} \bullet \\ \oplus \end{array} [H] \stackrel{(A)}{=} \text{---} \begin{array}{c} \bullet \\ \oplus \end{array} [H] \stackrel{(C)}{=} \text{---} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} [H]$$

$$\stackrel{(B)}{=} \text{---} \begin{array}{c} [Z(\pi/2)] \\ \oplus \end{array} \begin{array}{c} [Z(-\pi/2)] \\ \oplus \end{array} [H] \stackrel{(C)}{=} \begin{array}{c} [H] \\ \oplus \end{array} \begin{array}{c} [H] \\ \oplus \end{array} \begin{array}{c} [H] \\ \oplus \end{array} \stackrel{(A)}{=} \begin{array}{c} [H] \\ \oplus \end{array} \begin{array}{c} [H] \\ \oplus \end{array}$$

Circuit rewriting

Given a set of axioms,

$$\text{---} [H] [H] \text{---} \stackrel{(A)}{=} \text{---}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \bullet \end{array} \stackrel{(B)}{=} \begin{array}{c} \oplus \\ \bullet \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [H] \begin{array}{c} \oplus \\ \bullet \end{array} \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \bullet \end{array}$$

we can derive other equations. For instance,

$$\begin{array}{c} \bullet \\ \oplus \end{array} [H] \begin{array}{c} \oplus \\ \bullet \end{array} \stackrel{(A)}{=} \begin{array}{c} \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} [H] \begin{array}{c} \oplus \\ \bullet \end{array} \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \bullet \end{array} [H] \begin{array}{c} \oplus \\ \bullet \end{array}$$

$$\stackrel{(B)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \bullet \end{array} [H] \begin{array}{c} \oplus \\ \bullet \end{array} \stackrel{(C)}{=} \begin{array}{c} [H] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [H] \begin{array}{c} \oplus \\ \bullet \end{array} \stackrel{(A)}{=} \begin{array}{c} [H] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array}$$

Circuit rewriting

Given a set of axioms,

$$\text{---} [H] [H] \text{---} \stackrel{(A)}{=} \text{---}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(B)}{=} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [H] \begin{array}{c} \oplus \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array}$$

we can derive other equations. For instance,

$$\begin{array}{c} \bullet \\ \text{---} \end{array} [H] \begin{array}{c} \oplus \\ \text{---} \end{array} \stackrel{(A)}{=} \text{---} \begin{array}{c} \bullet \\ \text{---} \end{array} [H] \begin{array}{c} \oplus \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array} [H]$$

$$\stackrel{(B)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array} [H] \begin{array}{c} [H] \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} [H] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [H] [H] \begin{array}{c} [H] \\ \text{---} \end{array} \stackrel{(A)}{=} \begin{array}{c} [H] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array}$$

Circuit rewriting

Given a set of axioms,

$$\text{---} [H] [H] \text{---} \stackrel{(A)}{=} \text{---}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(B)}{=} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [H] \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array}$$

we can derive other equations. For instance,

$$\begin{array}{c} \bullet \\ \text{---} \end{array} [H] \stackrel{(A)}{=} \begin{array}{c} \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} [H] \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} [H] \\ \text{---} \end{array}$$

$$\stackrel{(B)}{=} \begin{array}{c} \text{---} [Z(\pi/2)] \oplus [Z(-\pi/2)] \oplus [H] \\ [H] [Z(\pi/2)] \bullet \bullet \end{array} \stackrel{(C)}{=} \begin{array}{c} [H] \oplus [H] [H] \\ [H] \bullet \end{array} \stackrel{(A)}{=} \begin{array}{c} [H] \oplus \\ [H] \bullet \end{array}$$

Circuit rewriting

Given a set of axioms,

$$\text{---} [H] [H] \text{---} \stackrel{(A)}{=} \text{---}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \bullet \end{array} \stackrel{(B)}{=} \begin{array}{c} \oplus \\ \bullet \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [H] \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [H] \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \bullet \end{array}$$

we can derive other equations. For instance,

$$\begin{array}{c} \bullet \\ \oplus \end{array} [H] \stackrel{(A)}{=} \begin{array}{c} \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} [H] \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \bullet \end{array} [H] \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array}$$

$$\stackrel{(B)}{=} \begin{array}{c} \text{---} \end{array} [Z(\pi/2)] \begin{array}{c} \oplus \\ \bullet \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \bullet \end{array} [H] \stackrel{(C)}{=} \begin{array}{c} [H] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} [H] \\ \text{---} \end{array} \begin{array}{c} [H] \\ \text{---} \end{array} \stackrel{(A)}{=} \begin{array}{c} [H] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \bullet \end{array}$$

Circuit rewriting

Given a set of axioms,

$$\text{---} [H] [H] \text{---} \stackrel{(A)}{=} \text{---}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \stackrel{(B)}{=} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(\alpha)] \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [H] \begin{array}{c} \oplus \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array}$$

we can derive other equations. For instance,

$$\begin{array}{c} \bullet \\ \text{---} \end{array} [H] \begin{array}{c} \oplus \\ \text{---} \end{array} \stackrel{(A)}{=} \begin{array}{c} \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} [H] \begin{array}{c} \oplus \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} [Z(\pi/2)] \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array} [H]$$

$$\stackrel{(B)}{=} \begin{array}{c} \text{---} \end{array} [Z(\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array} [Z(-\pi/2)] \begin{array}{c} \oplus \\ \text{---} \end{array} [H] \begin{array}{c} \oplus \\ \text{---} \end{array} \stackrel{(C)}{=} \begin{array}{c} [H] \oplus [H] [H] \\ \text{---} \end{array} \stackrel{(A)}{=} \begin{array}{c} [H] \oplus [H] \\ \text{---} \end{array}$$

Soundness and completeness

Is there a **set of axioms** $\mathcal{R}$ from which we can derive any true equation and only true equations in a graphical language $\mathbf{P}$?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \quad : \quad \mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket$$

Completeness

Any true equation is derivable.

$$\forall C_1, C_2 \quad : \quad \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket \implies \mathbf{P}/\mathcal{R} \vdash C_1 = C_2$$

[Clément,Heurtel,Mansfield,Perdrix,Valiron'2023]

The first **complete** and **sound** set of axioms for quantum circuits.

Soundness and completeness

Is there a **set of axioms** $\mathcal{R}$ from which we can derive any true equation and only true equations in a graphical language $\mathbf{P}$?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \quad : \quad \mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket$$

Completeness

Any true equation is derivable.

$$\forall C_1, C_2 \quad : \quad \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket \implies \mathbf{P}/\mathcal{R} \vdash C_1 = C_2$$

[Clément,Heurtel,Mansfield,Perdrix,Valiron'2023]

The first **complete** and **sound** set of axioms for quantum circuits.

Soundness and completeness

Is there a **set of axioms** $\mathcal{R}$ from which we can derive any true equation and only true equations in a graphical language $\mathbf{P}$?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \quad : \quad \mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket$$

Completeness

Any true equation is derivable.

$$\forall C_1, C_2 \quad : \quad \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket \implies \mathbf{P}/\mathcal{R} \vdash C_1 = C_2$$

[Clément,Heurtel,Mansfield,Perdrix,Valiron'2023]

The first **complete** and **sound** set of axioms for quantum circuits.

Soundness and completeness

Is there a **set of axioms** $\mathcal{R}$ from which we can derive any true equation and only true equations in a graphical language $\mathbf{P}$?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \quad : \quad \mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket$$

Completeness

Any true equation is derivable.

$$\forall C_1, C_2 \quad : \quad \llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket \implies \mathbf{P}/\mathcal{R} \vdash C_1 = C_2$$

[Clément,Heurtel,Mansfield,Perdrix,Valiron'2023]

The first **complete** and **sound** set of axioms for quantum circuits.

Complete and sound set of axioms

$$\begin{aligned}
 \textcircled{0} &= \boxed{} & \textcircled{2\pi} &= \boxed{} & \textcircled{\alpha_1} \textcircled{\alpha_2} &= \textcircled{\alpha_1 + \alpha_2} & -\boxed{H} \boxed{H} - &= - & -\boxed{Z(2\pi)} - &= - \\
 -\boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} - &= -\boxed{Z(\alpha_1 + \alpha_2)} - & -\boxed{X} \boxed{Z(\alpha)} \boxed{X} - &= \textcircled{\alpha} -\boxed{Z(-\alpha)} - \\
 -\boxed{H} - &= \textcircled{-\pi/4} -\boxed{Z(\pi/2)} \boxed{X(\pi/2)} \boxed{Z(\pi/2)} - & -\boxed{X(\alpha_1)} \boxed{Z(\alpha_2)} \boxed{X(\alpha_3)} - &= \textcircled{\beta_0} -\boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)} - \\
 \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} &= \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} & \boxed{Z(\alpha)} \begin{array}{c} \bullet \\ \oplus \end{array} &= \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\alpha)} & \text{X} &= \begin{array}{c} \bullet \oplus \bullet \\ \oplus \bullet \oplus \end{array} \\
 \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{X(\pi)} &= \boxed{X(\pi)} \begin{array}{c} \bullet \\ \oplus \end{array} & \boxed{H} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{H} &= \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(-\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \\
 \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} &= \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} & \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} &= \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \\
 \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(-\alpha_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{H} &= \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(-\alpha_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \\
 \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(-\alpha_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(-\alpha_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{H} &= \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(-\alpha_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(+\alpha_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(-\alpha_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{H} \\
 \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(\gamma_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(\gamma_4)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} &= \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{Z(\delta_1)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{Z(\delta_2)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(\delta_3)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{Z(\delta_5)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{X(\delta_6)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{Z(\delta_7)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array} \boxed{Z(\delta_9)} \begin{array}{c} \bullet \bullet \\ \oplus \oplus \end{array}
 \end{aligned}$$

Complete and sound set of axioms

$$\textcircled{2\pi} = \square$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$- \boxed{Z(\alpha_1)} - \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$\text{---}[H]\text{---} = \text{---}(-\pi/4)\text{---}[Z(\pi/2)]\text{---}[X(\pi/2)]\text{---}[Z(\pi/2)]\text{---}$$

The diagram shows a circuit with two horizontal lines. The top line has two black dots (control qubits) and the bottom line has two white circles with black outlines (target qubits). Vertical lines connect each control dot to its corresponding target circle, representing two CNOT gates. This is followed by an equals sign and a simplified circuit with two horizontal lines and no gates, representing the identity operation.

The diagram shows a crossing of two lines on the left, followed by an equals sign, and then a 2x2 grid of nodes on the right. The nodes are arranged in two rows and two columns. The top row has a solid black dot on the left and a circle with a plus sign on the right. The bottom row has a circle with a plus sign on the left and a solid black dot on the right. Vertical lines connect the top-left dot to the bottom-left plus, and the top-right plus to the bottom-right dot. Horizontal lines connect the top-left dot to the top-right plus, and the bottom-left plus to the bottom-right dot.

10/25

Complete and sound set of axioms

$$\textcircled{2\pi} = \square$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$- \boxed{Z(\alpha_1)} - \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$- [H] - = (-\pi/4) - [Z(\pi/2)] - [X(\pi/2)] - [Z(\pi/2)] -$$

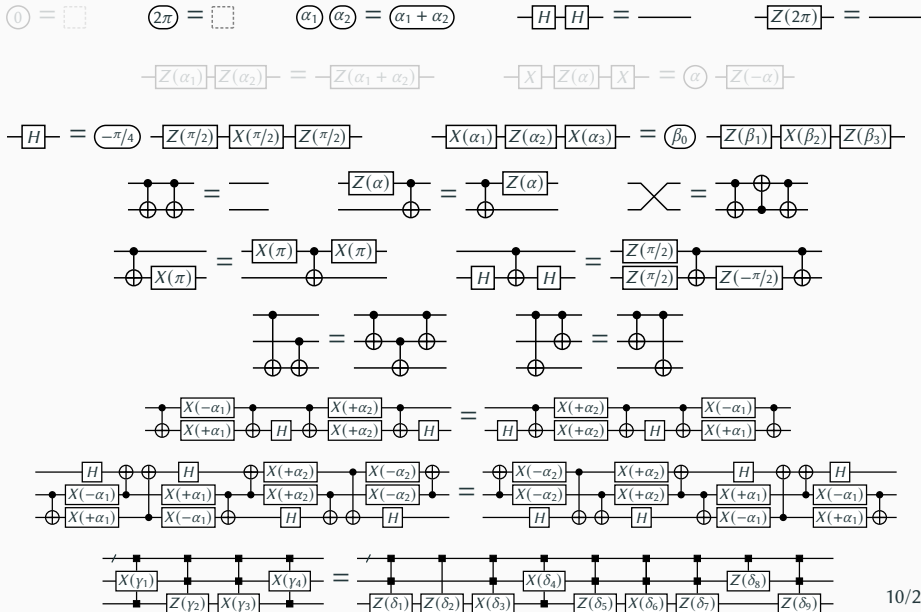
The diagram shows a circuit with two horizontal lines. The top line has two black dots (control qubits) and the bottom line has two white circles with black outlines (target qubits). Vertical lines connect each dot to its corresponding circle, representing two CNOT gates. This is followed by an equals sign and a simplified circuit with two horizontal lines and no gates, indicating that the two CNOT gates cancel each other out.

The diagram shows a crossing of two lines on the left, followed by an equals sign, and then a 2x2 grid of nodes on the right. The nodes are arranged in two rows and two columns. The top row has a solid black dot on the left and a circle with a plus sign on the right. The bottom row has a circle with a plus sign on the left and a solid black dot on the right. Vertical lines connect the top-left dot to the bottom-left plus, and the top-right plus to the bottom-right dot. Horizontal lines connect the top-left dot to the top-right plus, and the bottom-left plus to the bottom-right dot.

The diagram shows two equivalent quantum circuit representations. On the left, a CNOT gate (control on the top line, target on the bottom line) is followed by a phase gate $X(\pi)$ on the target line. This is shown to be equivalent to a phase gate $X(\pi)$ on the control line followed by a CNOT gate.

The diagram shows two equivalent quantum circuits. The left circuit has a control qubit (top line) and a target qubit (bottom line). The target qubit has Hadamard (H) gates before and after a CNOT gate controlled by the control qubit. The right circuit has the same control qubit, but the target qubit has $Z(\pi/2)$ gates before and after the CNOT gate, and an additional $Z(-\pi/2)$ gate after the CNOT gate. The two circuits are separated by an equals sign.

Complete and sound set of axioms



Complete and sound set of axioms

$$\textcircled{0} = \boxed{}$$

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$\boxed{Z(2\pi)} = \text{---}$$

$$\boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} = \boxed{Z(\alpha_1 + \alpha_2)}$$

$$\boxed{X} \boxed{Z(\alpha)} \boxed{X} = \textcircled{\alpha} \boxed{Z(-\alpha)}$$

$$\boxed{H} = \textcircled{-\pi/4} \boxed{Z(\pi/2)} \boxed{X(\pi/2)} \boxed{Z(\pi/2)}$$

$$\boxed{X(\alpha_1)} \boxed{Z(\alpha_2)} \boxed{X(\alpha_3)} = \textcircled{\beta_0} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)}$$

$$\text{---} \oplus \boxed{Z(\alpha)} \oplus \text{---} = \boxed{Z(\alpha)}$$

$$\text{---} \times \text{---} = \text{---} \oplus \oplus \text{---}$$

$$\text{---} \oplus \boxed{X(\pi)} = \boxed{X(\pi)} \text{---} \oplus \boxed{X(\pi)}$$

$$\boxed{H} \oplus \text{---} \oplus \boxed{H} = \boxed{Z(\pi/2)} \oplus \text{---} \oplus \boxed{Z(-\pi/2)}$$

$$\text{---} \oplus \oplus \text{---} = \text{---} \oplus \oplus \text{---}$$

$$\text{---} \oplus \oplus \text{---} = \text{---} \oplus \oplus \text{---}$$

$$\text{---} \oplus \boxed{X(-\alpha_1)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_1)} \oplus \text{---} = \text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{X(+\alpha_1)} \oplus \text{---} \oplus \boxed{X(-\alpha_1)}$$

$$\text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_2)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_2)} \oplus \text{---} = \text{---} \oplus \boxed{X(-\alpha_2)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{X(+\alpha_1)} \oplus \text{---} \oplus \boxed{X(-\alpha_1)}$$

$$\text{---} \oplus \boxed{X(\gamma_1)} \oplus \text{---} \oplus \boxed{X(\gamma_4)} \oplus \text{---} = \text{---} \oplus \boxed{Z(\delta_1)} \oplus \text{---} \oplus \boxed{Z(\delta_2)} \oplus \text{---} \oplus \boxed{X(\delta_3)} \oplus \text{---} \oplus \boxed{Z(\delta_5)} \oplus \text{---} \oplus \boxed{X(\delta_6)} \oplus \text{---} \oplus \boxed{Z(\delta_7)} \oplus \text{---} \oplus \boxed{Z(\delta_9)}$$

Complete and sound set of axioms

$$\textcircled{2\pi} = \square$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$- \boxed{Z(\alpha_1)} - \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$- [H] - = (-\pi/4) - [Z(\pi/2)] - [X(\pi/2)] - [Z(\pi/2)] -$$

[illegible]

10/25

Complete and sound set of axioms

$$\bigcirc = \boxed{}$$

$$2\pi = \boxed{}$$

$$\bigcirc\alpha_1 \bigcirc\alpha_2 = \bigcirc\alpha_1 + \alpha_2$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$\boxed{Z(2\pi)} = \text{---}$$

$$\boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} = \boxed{Z(\alpha_1 + \alpha_2)}$$

$$\boxed{X} \boxed{Z(\alpha)} \boxed{X} = \bigcirc\alpha \boxed{Z(-\alpha)}$$

$$\boxed{H} = \bigcirc-\pi/4 \boxed{Z(\pi/2)} \boxed{X(\pi/2)} \boxed{Z(\pi/2)}$$

$$\boxed{X(\alpha_1)} \boxed{Z(\alpha_2)} \boxed{X(\alpha_3)} = \bigcirc\beta_0 \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)}$$

$$\begin{array}{c} \bullet \\ \text{---} \end{array} \boxed{Z(\alpha)} \begin{array}{c} \bullet \\ \text{---} \end{array} = \boxed{Z(\alpha)} \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\text{X} = \begin{array}{c} \oplus \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{X(\pi)} = \boxed{X(\pi)} \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\boxed{H} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{H} = \boxed{Z(\pi/2)} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{Z(-\pi/2)} \begin{array}{c} \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \oplus \oplus \\ \text{---} \end{array} = \begin{array}{c} \oplus \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \oplus \oplus \\ \text{---} \end{array} = \begin{array}{c} \oplus \oplus \\ \text{---} \end{array}$$

$$\begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{X(-\alpha_1)} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_1)} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{H} = \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_1)} \begin{array}{c} \oplus \\ \text{---} \end{array} \boxed{X(-\alpha_1)}$$

$$\begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(-\alpha_2)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_1)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(-\alpha_1)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{H} = \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(-\alpha_2)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_2)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(+\alpha_1)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{X(-\alpha_1)} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{H} \begin{array}{c} \oplus \oplus \\ \text{---} \end{array} \boxed{H}$$

$$\begin{array}{c} \text{---} \end{array} \boxed{X(\gamma_1)} \begin{array}{c} \text{---} \end{array} \boxed{X(\gamma_4)} \begin{array}{c} \text{---} \end{array} \boxed{Z(\gamma_2)} \begin{array}{c} \text{---} \end{array} \boxed{X(\gamma_3)} = \begin{array}{c} \text{---} \end{array} \boxed{Z(\delta_1)} \begin{array}{c} \text{---} \end{array} \boxed{Z(\delta_2)} \begin{array}{c} \text{---} \end{array} \boxed{X(\delta_3)} \begin{array}{c} \text{---} \end{array} \boxed{Z(\delta_5)} \begin{array}{c} \text{---} \end{array} \boxed{X(\delta_6)} \begin{array}{c} \text{---} \end{array} \boxed{Z(\delta_7)} \begin{array}{c} \text{---} \end{array} \boxed{Z(\delta_9)}$$

Complete and sound set of axioms

$$\textcircled{2\pi} = \square$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$- \boxed{Z(\alpha_1)} - \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$- [H] - = (-\pi/4) - [Z(\pi/2)] - [X(\pi/2)] - [Z(\pi/2)] -$$

Complete and sound set of axioms

$$\textcircled{0} = \boxed{}$$

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$\boxed{Z(2\pi)} = \text{---}$$

$$\boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} = \boxed{Z(\alpha_1 + \alpha_2)}$$

$$\boxed{X} \boxed{Z(\alpha)} \boxed{X} = \textcircled{\alpha} \boxed{Z(-\alpha)}$$

$$\boxed{H} = \textcircled{-\pi/4} \boxed{Z(\pi/2)} \boxed{X(\pi/2)} \boxed{Z(\pi/2)}$$

$$\boxed{X(\alpha_1)} \boxed{Z(\alpha_2)} \boxed{X(\alpha_3)} = \textcircled{\beta_0} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)}$$

$$\text{---} \textcircled{\oplus} \boxed{Z(\alpha)} \textcircled{\oplus} \text{---} = \boxed{Z(\alpha)}$$

$$\text{---} \textcircled{\times} \text{---} = \text{---} \textcircled{\oplus} \text{---}$$

$$\text{---} \textcircled{\oplus} \boxed{X(\pi)} \text{---} = \boxed{X(\pi)} \text{---} \textcircled{\oplus} \text{---}$$

$$\boxed{H} \textcircled{\oplus} \boxed{H} = \boxed{Z(\pi/2)} \textcircled{\oplus} \boxed{Z(\pi/2)} \textcircled{\oplus} \boxed{Z(-\pi/2)}$$

$$\text{---} \textcircled{\oplus} \text{---} = \text{---} \textcircled{\oplus} \text{---}$$

$$\text{---} \textcircled{\oplus} \text{---} = \text{---} \textcircled{\oplus} \text{---}$$

$$\text{---} \textcircled{\oplus} \boxed{X(-\alpha_1)} \text{---} \text{---} \textcircled{\oplus} \boxed{X(+\alpha_2)} \text{---} = \text{---} \textcircled{\oplus} \boxed{X(+\alpha_2)} \text{---} \text{---} \textcircled{\oplus} \boxed{X(-\alpha_1)} \text{---}$$

$$\text{---} \textcircled{\oplus} \boxed{H} \text{---} \text{---} \textcircled{\oplus} \boxed{H} \text{---} \text{---} \textcircled{\oplus} \boxed{X(+\alpha_2)} \text{---} \text{---} \textcircled{\oplus} \boxed{X(-\alpha_2)} \text{---} = \text{---} \textcircled{\oplus} \boxed{X(-\alpha_2)} \text{---} \text{---} \textcircled{\oplus} \boxed{X(+\alpha_2)} \text{---} \text{---} \textcircled{\oplus} \boxed{H} \text{---} \text{---} \textcircled{\oplus} \boxed{H} \text{---}$$

$$\text{---} \textcircled{\oplus} \boxed{X(\gamma_1)} \text{---} \text{---} \textcircled{\oplus} \boxed{X(\gamma_4)} \text{---} = \text{---} \textcircled{\oplus} \boxed{Z(\delta_1)} \text{---} \text{---} \textcircled{\oplus} \boxed{Z(\delta_2)} \text{---} \text{---} \textcircled{\oplus} \boxed{X(\delta_3)} \text{---} \text{---} \textcircled{\oplus} \boxed{Z(\delta_5)} \text{---} \text{---} \textcircled{\oplus} \boxed{X(\delta_6)} \text{---} \text{---} \textcircled{\oplus} \boxed{Z(\delta_7)} \text{---} \text{---} \textcircled{\oplus} \boxed{Z(\delta_9)}$$

Complete and sound set of axioms

$$\textcircled{0} = \boxed{}$$

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$\boxed{Z(2\pi)} = \text{---}$$

$$\boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} = \boxed{Z(\alpha_1 + \alpha_2)}$$

$$\boxed{X} \boxed{Z(\alpha)} \boxed{X} = \textcircled{\alpha} \boxed{Z(-\alpha)}$$

$$\boxed{H} = \textcircled{-\pi/4} \boxed{Z(\pi/2)} \boxed{X(\pi/2)} \boxed{Z(\pi/2)}$$

$$\boxed{X(\alpha_1)} \boxed{Z(\alpha_2)} \boxed{X(\alpha_3)} = \textcircled{\beta_0} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)}$$

$$\text{---} \oplus \boxed{Z(\alpha)} \oplus \text{---} = \boxed{Z(\alpha)} \text{---}$$

$$\text{---} \times \text{---} = \text{---} \oplus \oplus \text{---}$$

$$\text{---} \oplus \boxed{X(\pi)} = \boxed{X(\pi)} \text{---} \oplus \boxed{X(\pi)} \text{---}$$

$$\boxed{H} \oplus \text{---} \oplus \boxed{H} = \boxed{Z(\pi/2)} \oplus \text{---} \oplus \boxed{Z(-\pi/2)} \oplus \text{---}$$

$$\text{---} \oplus \oplus \text{---} = \text{---} \oplus \oplus \text{---}$$

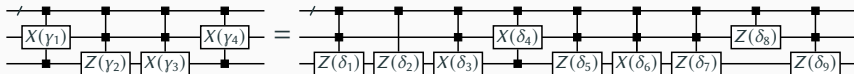
$$\text{---} \oplus \oplus \text{---} = \text{---} \oplus \oplus \text{---}$$

$$\text{---} \oplus \boxed{X(-\alpha_1)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_1)} \oplus \text{---} = \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_1)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_1)} \oplus \text{---}$$

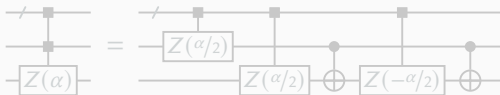
$$\text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_2)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{X(-\alpha_2)} \oplus \text{---} = \text{---} \oplus \boxed{X(-\alpha_2)} \oplus \text{---} \oplus \boxed{X(+\alpha_2)} \oplus \text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{H} \oplus \text{---} \oplus \boxed{X(+\alpha_1)} \oplus \text{---} \oplus \boxed{X(-\alpha_1)} \oplus \text{---}$$

$$\text{---} \oplus \boxed{X(\gamma_1)} \oplus \text{---} \oplus \boxed{X(\gamma_4)} \oplus \text{---} = \text{---} \oplus \boxed{Z(\delta_1)} \oplus \text{---} \oplus \boxed{Z(\delta_2)} \oplus \text{---} \oplus \boxed{X(\delta_3)} \oplus \text{---} \oplus \boxed{Z(\delta_5)} \oplus \text{---} \oplus \boxed{X(\delta_6)} \oplus \text{---} \oplus \boxed{Z(\delta_7)} \oplus \text{---} \oplus \boxed{Z(\delta_9)}$$

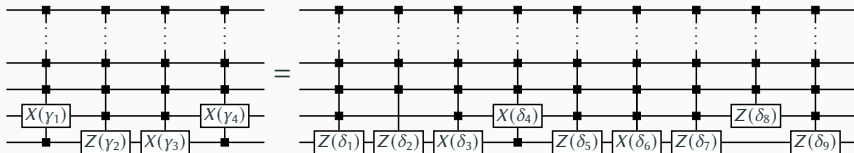
The huge and ugly axiom



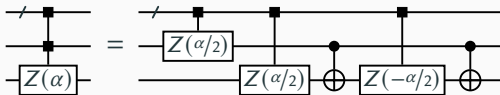
There is an instance of this axiom for any number of wires $n \geq 2$, and all gates are shorthand notations implementing controlled operations. For instance, the controlled version of $\boxed{Z(\alpha)}$ is defined as follows.



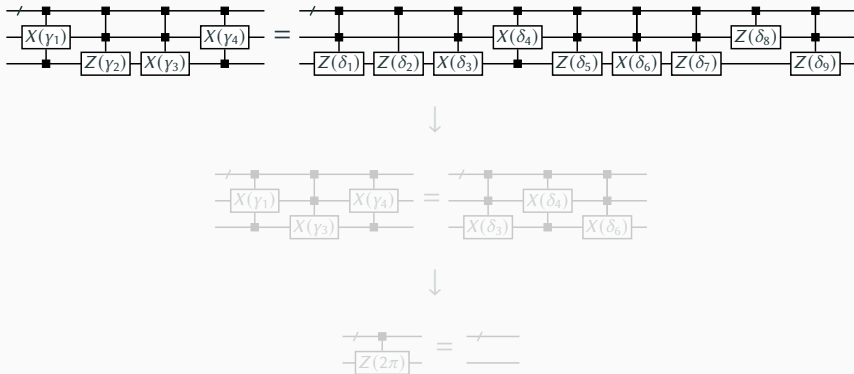
The huge and ugly axiom



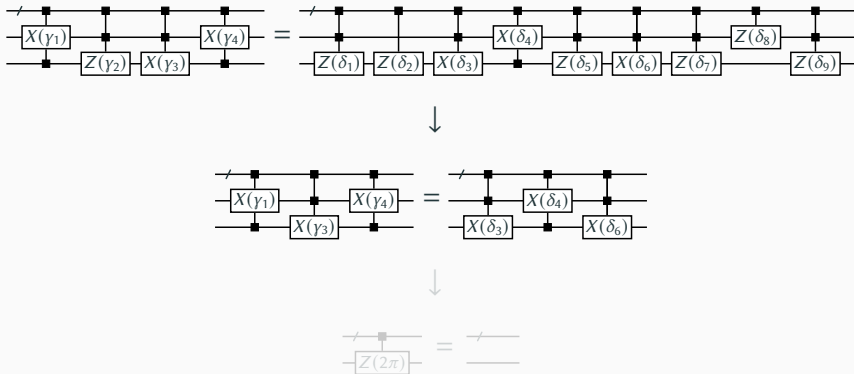
There is an instance of this axiom for any number of wires $n \geq 2$, and all gates are shorthand notations implementing controlled operations. For instance, the controlled version of $\boxed{Z(\alpha)}$ is defined as follows.



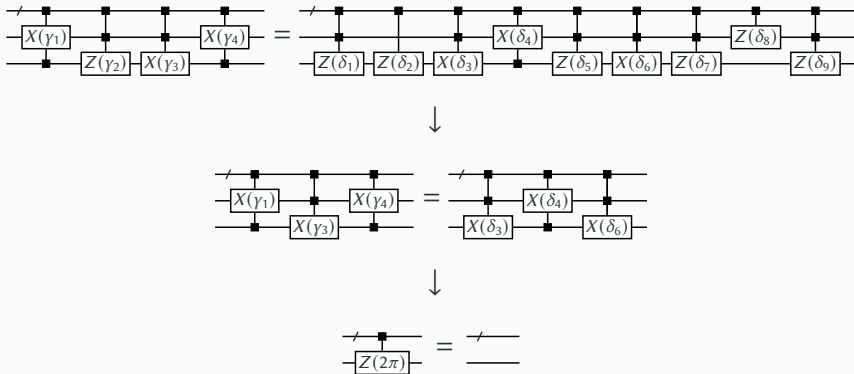
Huge, ugly, and useless...



Huge, ugly, and useless...



Huge, ugly, and useless...



New set of axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \boxed{Z(2\pi)} \text{---} = \text{---}$$

$$\text{---} \boxed{H} \text{---} = \textcircled{-\pi/4} \text{---} \boxed{Z(\pi/2)} \boxed{X(\pi/2)} \boxed{Z(\pi/2)} \text{---}$$

$$\text{---} \boxed{X(\alpha_1)} \boxed{Z(\alpha_2)} \boxed{X(\alpha_3)} \text{---} = \textcircled{\beta_0} \text{---} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)} \text{---}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \text{---} \boxed{Z(\alpha)} \text{---} \begin{array}{c} \bullet \\ \oplus \end{array} = \text{---} \boxed{Z(\alpha)} \text{---}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} = \begin{array}{c} \text{---} \bullet \\ \diagdown \quad \diagup \\ \text{---} \oplus \end{array}$$

$$\text{---} \begin{array}{c} \bullet \\ \oplus \end{array} \text{---} \boxed{H} \text{---} \begin{array}{c} \bullet \\ \oplus \end{array} \text{---} \boxed{H} \text{---} = \text{---} \boxed{Z(\pi/2)} \text{---} \begin{array}{c} \bullet \\ \oplus \end{array} \text{---} \boxed{Z(\pi/2)} \text{---} \begin{array}{c} \bullet \\ \oplus \end{array} \text{---} \boxed{Z(-\pi/2)} \text{---} \begin{array}{c} \bullet \\ \oplus \end{array} \text{---}$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \bullet \\ \text{---} \end{array} \boxed{Z(2\pi)} \text{---} = \text{---}$$

Theorem

The axioms are independent.

New set of axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$- \boxed{H} \boxed{H} - = -$$

$$- \boxed{Z(2\pi)} - = -$$

$$- \boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$- \boxed{X(\alpha_1)} \boxed{H} \boxed{X(\alpha_2)} - = \textcircled{\beta_0} - \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)} -$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\alpha)} \begin{array}{c} \bullet \\ \oplus \end{array} = \boxed{Z(\alpha)} -$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \oplus \begin{array}{c} \bullet \\ \oplus \end{array} = \begin{array}{c} \bullet \\ \oplus \end{array}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{H} = \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(-\pi/2)}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(2\pi)} - = -$$

Theorem

The axioms are independent.

New set of axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$- \boxed{H} \boxed{H} - = -$$

$$- \boxed{Z(2\pi)} - = -$$

$$- \boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$- \boxed{X(\alpha_1)} \boxed{H} \boxed{X(\alpha_2)} - = \textcircled{\beta_0} - \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)} -$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\alpha)} \begin{array}{c} \bullet \\ \oplus \end{array} = \boxed{Z(\alpha)} -$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \oplus \begin{array}{c} \bullet \\ \oplus \end{array} = \begin{array}{c} \bullet \\ \oplus \end{array}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{H} = \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(-\pi/2)}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(2\pi)} - = -$$

Theorem

The axioms are independent.

Really?

- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
- ...
- Let's cheat!

Really?

- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
- ...
- Let's cheat!

Really?

- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
- ...
- Let's cheat!

Really?

- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
- ...
- Let's cheat!

Really?

- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
- ...
- Let's cheat!

Really?

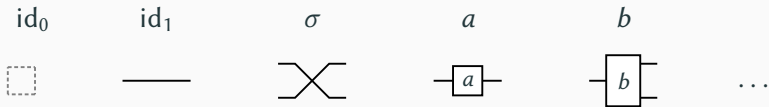
- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
- ...
- Let's cheat!

Really?

- What about the last axiom?
- Every instance is necessary. Every single of them.
- And what about Euler's axiom?
- You need at least uncountably many instances of Euler's axiom.
- And with a different set of generators?
- Same issue. I told you, we can no longer improve it. It's over!
- Isn't that sad?
- That's the way it is.
- ...
- ...
- Let's cheat!

**“If the game is impossible,
then change the rules.”**

Generators



Composition operators

$$\frac{f : n \rightarrow p \quad g : p \rightarrow m}{f \circ g : n \rightarrow m}$$



$$\frac{f : n \rightarrow m \quad g : p \rightarrow q}{f \otimes g : n + p \rightarrow m + q}$$



NEW!

$$\frac{f : n \rightarrow n}{C(f) : 1 + n \rightarrow 1 + n}$$



Composition operators

$$\frac{f : n \rightarrow p \quad g : p \rightarrow m}{f \circ g : n \rightarrow m}$$



$$\frac{f : n \rightarrow m \quad g : p \rightarrow q}{f \otimes g : n + p \rightarrow m + q}$$

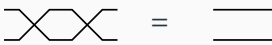


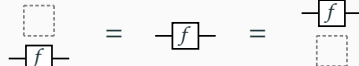
NEW!

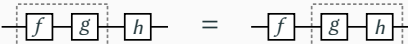
$$\frac{f : n \rightarrow n}{C(f) : 1 + n \rightarrow 1 + n}$$

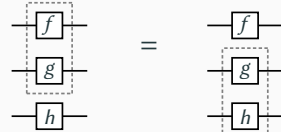


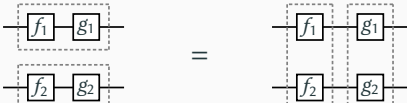
Coherence equations

$$\sigma \circ \sigma = \text{id}_1 \otimes \text{id}_1$$


$$\text{id}_0 \otimes f = f = f \otimes \text{id}_0$$


$$(f \circ g) \circ h = f \circ (g \circ h)$$


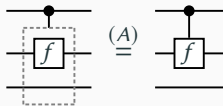
$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$


$$(f_1 \circ g_2) \otimes (f_2 \circ g_2) = (f_1 \otimes f_2) \circ (g_1 \otimes g_2)$$


...

Coherence equations

NEW!



NEW!

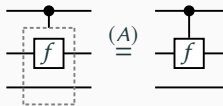


NEW!

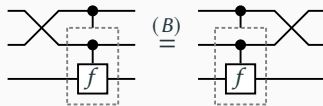


Coherence equations

NEW!



NEW!

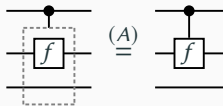


NEW!

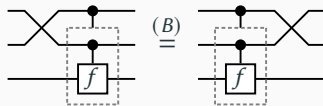


Coherence equations

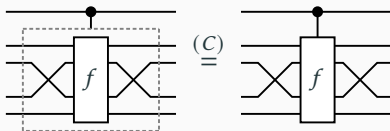
NEW!



NEW!



NEW!



Controlled PROPs

Definition

A **control functor** maps any endocircuit $f : n \rightarrow n$ to the endocircuit $C(f) : (1 + n) \rightarrow (1 + n)$ while satisfying (A), (B), and (C).

Definition

A **controlled PROP** is a PROP equipped with a control functor.

Controlled PROPs

Definition

A **control functor** maps any endocircuit $f : n \rightarrow n$ to the endocircuit $C(f) : (1 + n) \rightarrow (1 + n)$ while satisfying (A), (B), and (C).

Definition

A **controlled PROP** is a PROP equipped with a control functor.

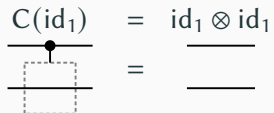
Controlled PROPs

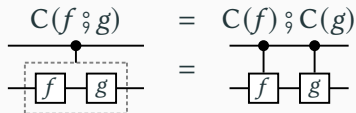
$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

$$C(f \circ g) = C(f) \circ C(g)$$

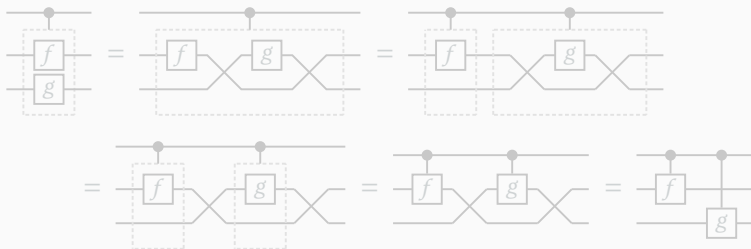
Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Controlled PROPs

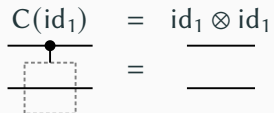
$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$


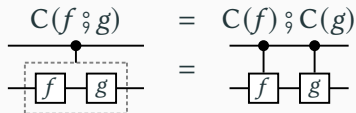
$$C(f \circ g) = C(f) \circ C(g)$$


Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

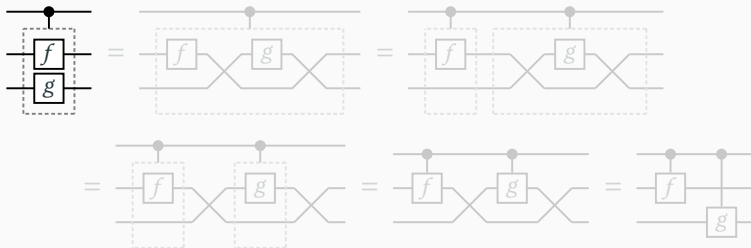


Controlled PROPs

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$


$$C(f \circ g) = C(f) \circ C(g)$$


Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have



Controlled PROPs

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

$$C(f \circ g) = C(f) \circ C(g)$$

Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Controlled PROPs

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

$$C(f \circ g) = C(f) \circ C(g)$$

Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Controlled PROPs

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

$$C(f \circ g) = C(f) \circ C(g)$$

Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Controlled PROPs

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

$$C(f \circ g) = C(f) \circ C(g)$$

Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Controlled PROPs

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

$$C(f \circ g) = C(f) \circ C(g)$$

Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Does anyone actually read the headlines anyway?

$$\mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \mathbf{P}/\mathcal{R} \vdash C(C_1) = C(C_2) \quad \text{NEW!}$$

Example

Let **CNot** be the controlled PROP generated by $\text{---}\oplus\text{---}$ with the following axioms.

$$\text{---}\oplus\text{---}\oplus\text{---} = \text{---}, \quad \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \oplus \quad \oplus \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \oplus \quad \oplus \\ \bullet \end{array}, \quad \begin{array}{c} \oplus \quad \bullet \quad \oplus \\ \diagdown \quad \diagup \\ \oplus \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \oplus \quad \oplus \end{array}.$$

Then, $\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \oplus \end{array}$ is not a generator. It is the circuit $C(\text{---}\oplus\text{---})$.

Moreover, the equations is $\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \oplus \quad \oplus \end{array} = \text{---}$ is derivable.

Does anyone actually read the headlines anyway?

$$\mathbf{P}/\mathcal{R} \vdash C_1 = C_2 \implies \mathbf{P}/\mathcal{R} \vdash C(C_1) = C(C_2) \quad \text{NEW!}$$

Example

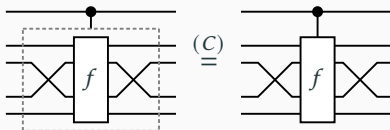
Let **CNot** be the controlled PROP generated by $\text{---}\oplus\text{---}$ with the following axioms.

$$\text{---}\oplus\text{---}\oplus\text{---} = \text{---}, \quad \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \oplus \quad \oplus \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \oplus \quad \oplus \\ \bullet \end{array}, \quad \begin{array}{c} \oplus \quad \bullet \quad \oplus \\ \diagdown \quad \diagup \\ \oplus \quad \oplus \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \oplus \quad \oplus \end{array}.$$

Then, $\begin{array}{c} \bullet \\ \diagdown \\ \oplus \end{array}$ is not a generator. It is the circuit $C(\text{---}\oplus\text{---})$.

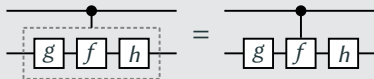
Moreover, the equations $\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \oplus \quad \oplus \end{array} = \text{---}$ is derivable.

Conjugated PROP



Definition

A **conjugated PROP** is a controlled PROP such that



is satisfied whenever $\mathbf{P}/\mathcal{R} \vdash \text{---}[g][h]\text{---} = \text{---}$.

Simpler generators



,



,



,



Simpler generators



,



,



,



Simpler generators



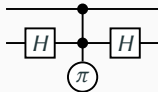
,



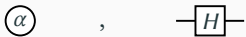
,



,



Simpler generators



Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$- \boxed{H} \boxed{H} - = \text{---}$$

$$- \boxed{Z(2\pi)} - = \text{---}$$

$$- \boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} - = - \boxed{Z(\alpha_1 + \alpha_2)} -$$

$$- \boxed{X(\alpha_1)} \boxed{H} \boxed{X(\alpha_2)} - = \textcircled{\beta_0} - \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)} -$$

$$\begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{Z(\alpha)} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} = \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{Z(\alpha)} \begin{array}{c} \bullet \\ | \\ \oplus \end{array}$$

$$\begin{array}{c} \oplus \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} = \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{H} = \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{Z(-\pi/2)} \begin{array}{c} \bullet \\ | \\ \oplus \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \oplus \end{array} \boxed{Z(2\pi)} \begin{array}{c} \bullet \\ | \\ \oplus \end{array} = \begin{array}{c} \bullet \\ | \\ \oplus \end{array} \begin{array}{c} \bullet \\ | \\ \oplus \end{array}$$

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \text{---} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \bullet \text{---} = \text{---}$$

$\textcircled{2\pi}$

$$\begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \bullet \\ \text{---} \end{array}$$

$\textcircled{\alpha_1} \quad \textcircled{\alpha_2} \qquad \textcircled{\alpha_1 + \alpha_2}$

$$\text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---} \boxed{H} \text{---} \bullet \text{---}$$

$\textcircled{\alpha_1} \quad \textcircled{\alpha_2} \qquad \textcircled{\beta_1} \quad \textcircled{\beta_2} \quad \textcircled{\beta_3}$

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \text{---} \quad \text{---} \quad \text{---} \end{array} = \begin{array}{c} \bullet \\ \text{---} \end{array}$$

$\textcircled{\pi} \quad \textcircled{\alpha} \quad \textcircled{\pi} \qquad \textcircled{\alpha}$

$$\text{X} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \text{---} \quad \text{---} \quad \text{---} \end{array}$$

$\textcircled{\pi} \quad \textcircled{\pi} \quad \textcircled{\pi}$

$$\begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \end{array}$$

$\textcircled{\pi} \qquad \textcircled{\pi/2} \quad \textcircled{\pi/2} \quad \textcircled{\pi} \quad \textcircled{-\pi/2} \quad \textcircled{\pi}$

$$\text{---} \bullet \text{---} = \text{---}$$

$\textcircled{2\pi}$

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \text{---} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \textcircled{2\pi} \text{---} = \text{---}$$

$$\begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \textcircled{\alpha_1} \quad \textcircled{\alpha_2} \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\alpha_1 + \alpha_2} \end{array}$$

$$\text{---} \boxed{H} \text{---} \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\alpha_1} \end{array} \boxed{H} \text{---} \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\alpha_2} \end{array} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\beta_1} \end{array} \boxed{H} \text{---} \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\beta_2} \end{array} \boxed{H} \text{---} \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\beta_3} \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \text{---} \quad \text{---} \quad \text{---} \\ \textcircled{\pi} \quad \textcircled{\alpha} \quad \textcircled{\pi} \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\alpha} \end{array}$$

$$\text{X} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \text{---} \quad \text{---} \quad \text{---} \\ \boxed{H} \quad \boxed{H} \quad \boxed{H} \\ \text{---} \quad \text{---} \quad \text{---} \\ \textcircled{\pi} \quad \textcircled{\pi} \quad \textcircled{\pi} \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\pi} \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \textcircled{\pi/2} \quad \textcircled{\pi/2} \quad \textcircled{\pi} \quad \textcircled{-\pi/2} \quad \textcircled{\pi} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \boxed{H} \quad \boxed{H} \quad \boxed{H} \quad \boxed{H} \end{array}$$

$$\text{---} \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{2\pi} \end{array} \text{---} = \text{---}$$

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1 + \alpha_2}}{\mid}} \text{---}$$

$$\text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \overset{\bullet}{\underset{\textcircled{\beta_1}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_2}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_3}}{\mid}} \text{---}$$

$$\begin{array}{c} \bullet & \bullet & \bullet \\ \hline \bullet & & \bullet \\ \hline \textcircled{\pi} & \textcircled{\alpha} & \textcircled{\pi} \end{array} = \begin{array}{c} \bullet \\ \hline \bullet \\ \hline \textcircled{\alpha} \end{array}$$

$$\text{X} = \begin{array}{c} \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & & \bullet & \bullet \\ \hline \textcircled{\pi} & \textcircled{\pi} & \textcircled{\pi} & \textcircled{\pi} \end{array} \begin{array}{c} \boxed{H} & \boxed{H} & \boxed{H} & \boxed{H} \\ \hline \textcircled{\pi} & \textcircled{\pi} & \textcircled{\pi} & \textcircled{\pi} \end{array}$$

$$\begin{array}{c} \bullet \\ \hline \bullet \\ \hline \textcircled{\pi} \end{array} = \begin{array}{c} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \textcircled{\pi/2} & \textcircled{\pi/2} & \textcircled{\pi} & \textcircled{-\pi/2} & \textcircled{\pi} & \textcircled{\pi} \end{array} \begin{array}{c} \boxed{H} & \boxed{H} & \boxed{H} & \boxed{H} & \boxed{H} & \boxed{H} \\ \hline \textcircled{\pi/2} & \textcircled{\pi/2} & \textcircled{\pi} & \textcircled{-\pi/2} & \textcircled{\pi} & \textcircled{\pi} \end{array}$$

$$\begin{array}{c} \bullet \\ \hline \bullet \\ \hline \textcircled{2\pi} \end{array} = \text{---}$$

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\alpha_1 \alpha_2 = \alpha_1 + \alpha_2$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$\text{---} \bullet \text{---} = \text{---}$$

$$\begin{array}{c} \text{---}[H]\text{---} \bullet \text{---}[H]\text{---} \bullet \text{---}[H]\text{---} \\ \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ \quad \quad \quad \alpha_1 \quad \quad \quad \alpha_2 \end{array} = \begin{array}{c} \text{---}(\beta_0)\text{---} \bullet \text{---}[H]\text{---} \bullet \text{---}[H]\text{---} \bullet \text{---} \\ \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ \quad \quad \quad \beta_1 \quad \quad \quad \beta_2 \quad \quad \quad \beta_3 \end{array}$$

The diagram shows an equality between two configurations of lines and circles. On the left, a horizontal line has a black dot. Below it, a dashed rectangle encloses three circles labeled π , α , and π from left to right. Each circle is connected to the line above it by a vertical line segment. On the right, the same horizontal line has a black dot, but only one circle labeled α is connected to it by a vertical line segment. This represents the fusion of three pions into an alpha particle.

The diagram shows an equality between two circuit configurations. On the left, a fermion line (indicated by a slash) and a 2π phase (a circle with 2π) are connected by a vertical line, with dots at the connection points. On the right, the fermion line is crossed over the 2π phase. This represents the braid relation for a fermion and a full phase rotation.

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$- \boxed{H} \boxed{H} - = -$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1 + \alpha_2}}{\mid}} \text{---}$$

$$- \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} - \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} - \boxed{H} - = \textcircled{\beta_0} - \overset{\bullet}{\underset{\textcircled{\beta_1}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_2}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_3}}{\mid}} -$$

$$\text{---} \overset{\bullet}{\underset{\text{---} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \textcircled{\alpha} \text{---} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha}}{\mid}} \text{---}$$

$$\bowtie = \text{---} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \text{---}$$

$$\text{---} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \text{---} = \text{---} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \boxed{H} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \text{---}$$

$$\text{---} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \text{---} = \text{---} \overset{\bullet}{\mid} \text{---} \overset{\bullet}{\mid} \text{---}$$

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1 + \alpha_2}}{\mid}} \text{---}$$

$$\text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \overset{\bullet}{\underset{\textcircled{\beta_1}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_2}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_3}}{\mid}} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\text{---} \textcircled{\pi} \textcircled{\alpha} \textcircled{\pi} \text{---}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha}}{\mid}} \text{---}$$

$$\text{---} \times \text{---} = \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\pi/2}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\pi/2}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{-\pi/2}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1 + \alpha_2}}{\mid}} \text{---}$$

$$\text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \overset{\bullet}{\underset{\textcircled{\beta_1}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_2}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_3}}{\mid}} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\text{---} \textcircled{\pi} \textcircled{\alpha} \textcircled{\pi} \text{---}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha}}{\mid}} \text{---}$$

$$\text{---} \times \text{---} = \text{---} \overset{\bullet}{\underset{\boxed{H}}{\mid}} \text{---} \overset{\bullet}{\underset{\boxed{H}}{\mid}} \text{---} \overset{\bullet}{\underset{\boxed{H}}{\mid}} \text{---} \overset{\bullet}{\underset{\boxed{H}}{\mid}} \text{---} \overset{\bullet}{\underset{\boxed{H}}{\mid}} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\pi/2}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\pi/2}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{-\pi/2}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

Simpler axioms

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha_1 + \alpha_2}}{\mid}} \text{---}$$

$$\text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_1}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\alpha_2}}{\mid}} \text{---} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \overset{\bullet}{\underset{\textcircled{\beta_1}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_2}}{\mid}} \boxed{H} \overset{\bullet}{\underset{\textcircled{\beta_3}}{\mid}} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\text{---} \textcircled{\pi} \textcircled{\alpha} \textcircled{\pi} \text{---}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\alpha}}{\mid}} \text{---}$$

$$\text{---} \times \text{---} = \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} = \text{---} \overset{\bullet}{\underset{\textcircled{\pi/2}}{\mid}} \text{---} \overset{\bullet}{\underset{\textcircled{\pi/2}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{-\pi/2}}{\mid}} \text{---} \boxed{H} \overset{\bullet}{\underset{\textcircled{\pi}}{\mid}} \text{---} \boxed{H} \text{---}$$

$$\text{---} \overset{\bullet}{\underset{\textcircled{2\pi}}{\mid}} \text{---} = \text{---}$$

Final set of axioms using controlled PROPs

$$(2\pi) = \square \quad (\alpha_1)(\alpha_2) = (\alpha_1 + \alpha_2)$$

$$\begin{array}{c} \text{---} [H] \bullet \text{---} [H] \bullet \text{---} [H] \text{---} \\ \quad \quad \quad \alpha_1 \quad \quad \quad \alpha_2 \end{array} = (\beta_0) \begin{array}{c} \text{---} \bullet \text{---} [H] \bullet \text{---} [H] \bullet \text{---} \\ \quad \quad \quad \beta_1 \quad \quad \quad \beta_2 \quad \quad \quad \beta_3 \end{array}$$

$$\text{---} [H] [H] \text{---} = \text{---}$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \text{---} \bullet \text{---} [H] \bullet \text{---} [H] \bullet \text{---} \\ | \quad | \quad | \\ [H] \bullet \text{---} [H] \bullet \text{---} [H] \bullet \text{---} [H] \\ \pi \quad \pi \quad \pi \end{array}$$

Final set of axioms using controlled PROPs

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \boxed{X(\alpha_1)} \boxed{H} \boxed{X(\alpha_2)} \text{---} = \textcircled{\beta_0} \text{---} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)} \text{---}$$

$$\text{---} \times \text{---} = \begin{array}{c} \bullet \oplus \bullet \\ | \quad | \\ \oplus \bullet \oplus \end{array}$$

Final set of axioms using controlled PROPs

$$(2\pi) = \square$$

$$(\alpha_1)(\alpha_2) = (\alpha_1 + \alpha_2)$$

$$-H-H- = -$$

$$-X(\alpha_1)-H-X(\alpha_2)- = (\beta_0) -Z(\beta_1)-X(\beta_2)-Z(\beta_3)-$$

$$\bowtie = \begin{array}{c} \bullet \oplus \bullet \\ | \quad | \\ \oplus \bullet \oplus \end{array}$$

$$\left(\begin{array}{ccc} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ -H-Z(\alpha)-H- \end{array} = \begin{array}{c} \bullet \\ | \\ -H-Z(\alpha)-H- \end{array} & \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ -Z(\alpha)-H-Z(\alpha)- \end{array} = \begin{array}{c} \bullet \\ | \\ -Z(\alpha)-H-Z(\alpha)- \end{array} \\ \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ -H-Z(\pi)-H- \end{array} = \begin{array}{c} \bullet \\ | \\ -H-Z(\pi)-H- \end{array} & \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ -Z(\pi)-H-Z(\pi)- \end{array} = \begin{array}{c} \bullet \\ | \\ -Z(\pi)-H-Z(\pi)- \end{array} \end{array} \right)$$

Thanks



(arXiv:2508.21756)

**Diagrammatic Reasoning with Control as a Constructor,
Applications to Quantum Circuits**