Towards Term-based Verification of Diagrammatic Equivalence

IJCAR 2026

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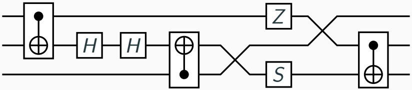
Quantum circuits



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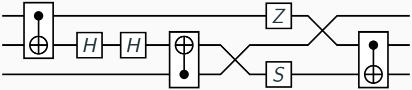

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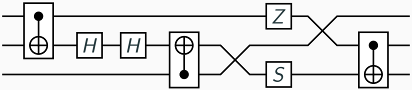

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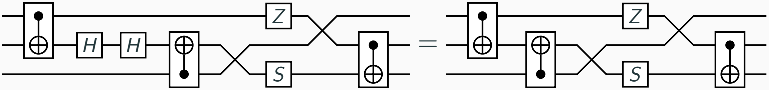

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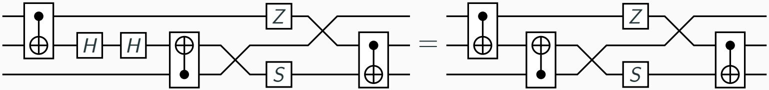

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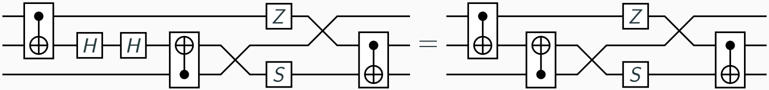

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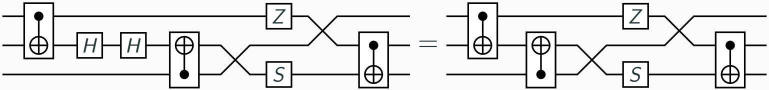

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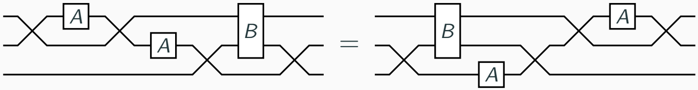
Crucial task in quantum compilation pipelines.

Our long-term goal: Certify quantum circuit equivalences.

String diagrams



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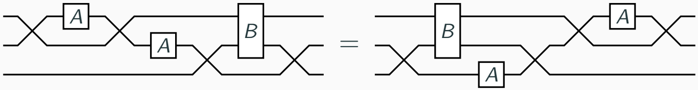
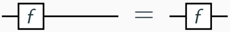
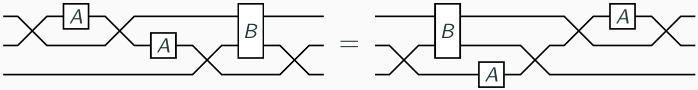


The string diagram formalism comes with a bunch of coherence laws governing the intuitive behavior of wires and boxes.





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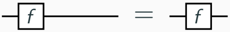

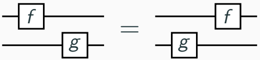

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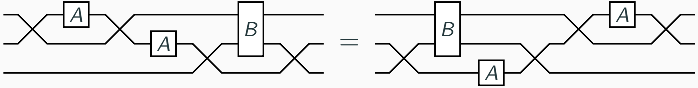

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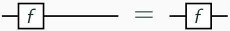

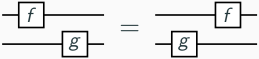

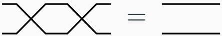

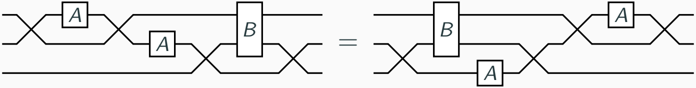

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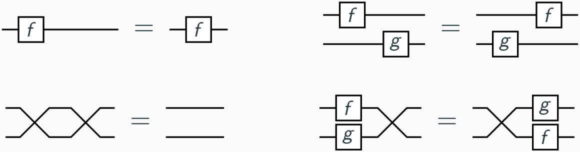

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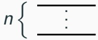
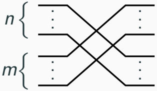
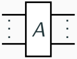
From diagrams to terms



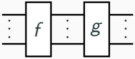
$\longrightarrow \mathrm{id}_n$



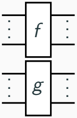
$\longrightarrow \sigma_{n,m}$



$\longrightarrow A$

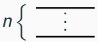


$\longrightarrow f \fatsemi g$



$\longrightarrow f \otimes g$

From diagrams to terms



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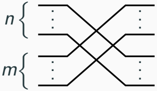

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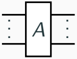

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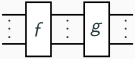

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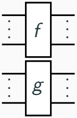
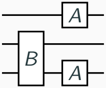


$\longrightarrow f \otimes g$

But then, there are multiple terms representing the same diagram.



$$\longrightarrow \begin{cases} A \otimes (B \mathbin{;} (\mathrm{id}_1 \otimes A)) \\ (\mathrm{id}_1 \otimes B) \mathbin{;} (A \otimes (\mathrm{id}_1 \otimes A)) \\ (A \otimes B) \mathbin{;} (\mathrm{id}_2 \otimes A) \\ \ldots \end{cases}$$

Coherence laws as terms

The coherence laws yield rewrite equations between terms.

$$f \fatsemi \mathrm{id}_n = f$$

$$(f \otimes \mathrm{id}_n) \fatsemi (\mathrm{id}_m \otimes g) = (\mathrm{id}_m \otimes g) \fatsemi (f \otimes \mathrm{id}_n)$$

$$\sigma_{n,m} \fatsemi \sigma_{m,n} = \mathrm{id}_{n+m}$$

$$(f \otimes g) \fatsemi \sigma_{m,q} = \sigma_{n,p}(g \otimes f)$$

Diagrammatic equivalence

Given two diagrammatically equivalent terms t_1 and t_2, compute a rewrite sequence from t_1 to t_2.

Contributions

General problem seems difficult. We solve two sub-problems.

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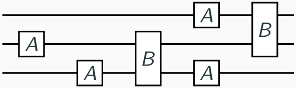
(1) Diagrams without swaps.



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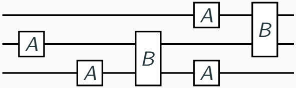
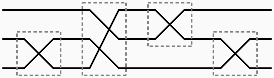


(2) Diagrams with only swaps.



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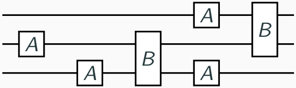

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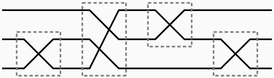

We tried some automated tools to encode the problem (egglog, Vampire, E, etc.). But it was not successful, especially because of the inherent presence of arithmetic in the problem.

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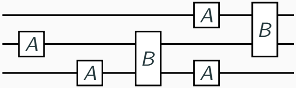

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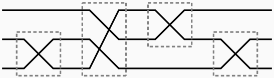

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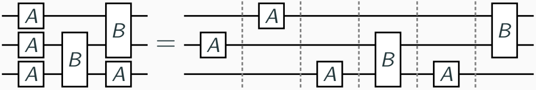
Instead, we design terminating and confluent term rewriting systems.

Diagrams without swaps

Sequence of layers

Idea: Transform diagrams as sequences of layers.



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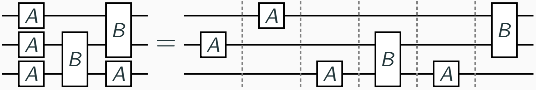
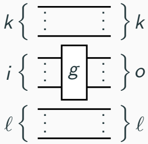
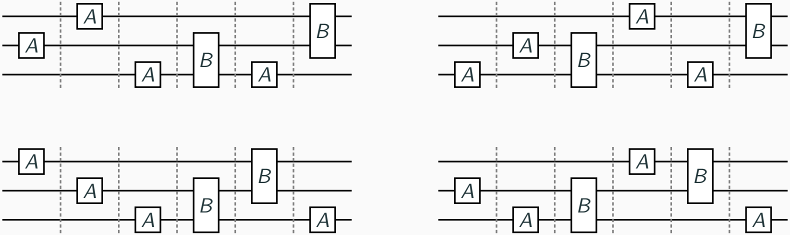


$L^k_\ell(g) := \mathrm{id}_k \otimes (g \otimes \mathrm{id}_\ell) =$ 

Ordering the layers

Issue: How to order the layers?



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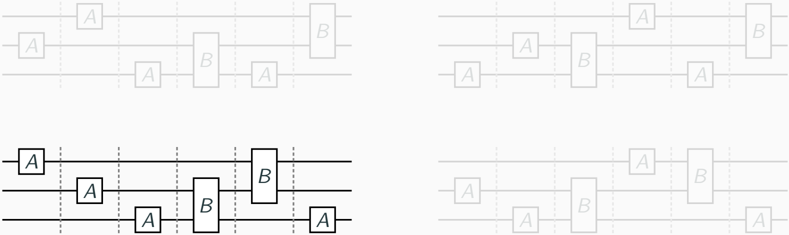
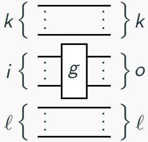


Normal form

$$L_\ell^k(g) \coloneqq \mathrm{id}_k \otimes (g \otimes \mathrm{id}_\ell) =$$



Definition

A normal form is a sequence of layers

$$L_{\ell_1}^{k_1}(g_1) \fatsemi \left(L_{\ell_2}^{k_2}(g_2) \fatsemi \left(L_{\ell_3}^{k_3}(g_3) \fatsemi \left(\dots \right) \right) \right)$$

such that

$$k_1 < k_2 + i_2 \quad , \quad k_2 < k_3 + i_3 \quad , \quad k_3 < k_4 + i_4 \quad , \quad \dots$$

Rewriting system

$$(t \mathbin{⨟} u) \mathbin{⨟} v \longrightarrow t \mathbin{⨟} (u \mathbin{⨟} v) \quad \text{(R1)} \qquad (t \otimes u) \otimes v \longrightarrow t \otimes (u \otimes v) \quad \text{(R2)}$$

$$\mathrm{id}_n \mathbin{⨟} t \longrightarrow t \quad \text{(R3)} \qquad t \mathbin{⨟} \mathrm{id}_n \longrightarrow t \quad \text{(R4)}$$

$$\mathrm{id}_n \otimes \mathrm{id}_m \longrightarrow \mathrm{id}_{n+m} \quad \text{(R5)} \qquad \mathrm{id}_n \otimes (\mathrm{id}_m \otimes t) \longrightarrow \mathrm{id}_{n+m} \otimes t \quad \text{(R6)}$$

$$\begin{array}{c} u \otimes v \longrightarrow (u \otimes \mathrm{id}_{\mathrm{dom}(v)}) \mathbin{⨟} (\mathrm{id}_{\mathrm{cod}(u)} \otimes v) \\ \text{if } \; u \neq \mathrm{id}_n \wedge v \neq \mathrm{id}_m \end{array} \quad \text{(R7)}$$

$$\mathrm{id}_n \otimes (u \mathbin{⨟} v) \longrightarrow (\mathrm{id}_n \otimes u) \mathbin{⨟} (\mathrm{id}_n \otimes v) \quad \text{(R8)}$$

$$(u \mathbin{⨟} v) \otimes \mathrm{id}_n \longrightarrow (u \otimes \mathrm{id}_n) \mathbin{⨟} (v \otimes \mathrm{id}_n) \quad \text{(R9)}$$

$$\begin{array}{c} L^{k_1}_{\ell_1}(g_1) \mathbin{⨟} L^{k_2}_{\ell_2}(g_2) \longrightarrow L^{k_2}_{\ell_2+i_1-o_1}(g_2) \mathbin{⨟} L^{k_1-i_2+o_2}_{\ell_1}(g_1) \\ \text{if } \; k_1 \geq k_2 + i_2 \end{array} \quad \text{(R10)}^*$$

$*$ There are actually two instances of this rule, allowing to apply it in a bigger sequence of layers.

Result

Theorem

The rewriting system is terminating and confluent.

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$$
\begin{tikzcd}
(A \otimes B) \fatsemi (\mathrm{id}_2 \otimes A) \arrow[dr] & \simeq & A \otimes (B \fatsemi (\mathrm{id}_1 \otimes A)) \arrow[dl] \\
& (\mathrm{id}_0 \fatsemi (A \otimes \mathrm{id}_2)) \fatsemi ((\mathrm{id}_1 \fatsemi (B \otimes \mathrm{id}_0)) \fatsemi (\mathrm{id}_2 \fatsemi (A \otimes \mathrm{id}_0))) &
\end{tikzcd}
$$

$$
= L_2^0(A) \fatsemi (L_0^1(B) \fatsemi L_0^2(A))
$$

Result

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\end{tikzcd}
$$

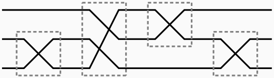
$$= L^0_2(A) \fatsemi (L^1_0(B) \fatsemi L^2_0(A))$$

The key steps of the proofs have been verified using the proof assistant Isabelle/HOL.

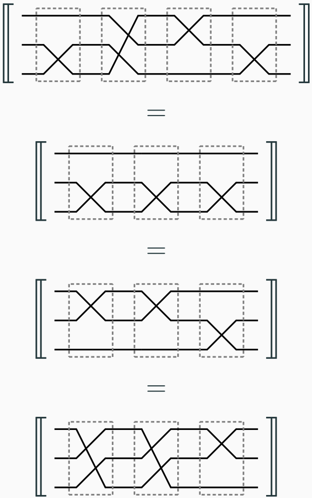
Diagrams with only swaps

Permutations

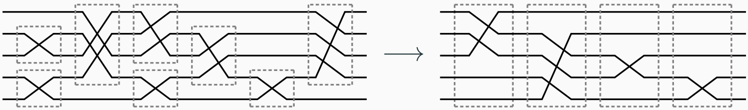
$$[\![-]\!] : \text{terms} \to \text{permutations}$$

$[\![$  $]\!] = (1, 3, 2)$

The normal form is no longer unique



Canonical form



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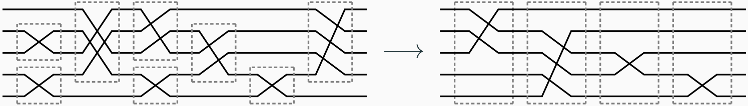


Definition

A canonical form is a normal form

$$L^{k_1}_{\ell_1}(\sigma_{n_1,m_1}) \fatsemi \left(L^{k_2}_{\ell_2}(\sigma_{n_2,m_2}) \fatsemi \left(L^{k_3}_{\ell_3}(\sigma_{n_3,m_3}) \fatsemi \left(\dots \right)\right)\right)$$

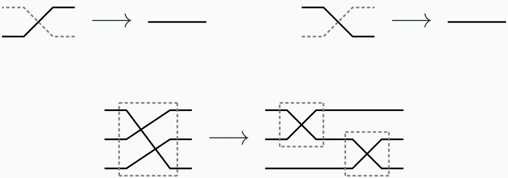
such that $m_1 = m_2 = m_3 = \cdots = 1$ and

$$k_1 < k_2 \quad , \quad k_2 < k_3 \quad , \quad k_3 < k_4 \quad , \quad \dots$$

New rewrite rules

$$L^{k_1}_{\ell_1}(\sigma_{n_1,m_1}) \fatsemi \left(L^{k_2}_{\ell_2}(\sigma_{n_2,m_2}) \fatsemi \left(L^{k_3}_{\ell_3}(\sigma_{n_3,m_3}) \fatsemi \left(\dots \right)\right)\right)$$

The conditions $m_i = 1$ are satisfied with the following new rewrite rules.



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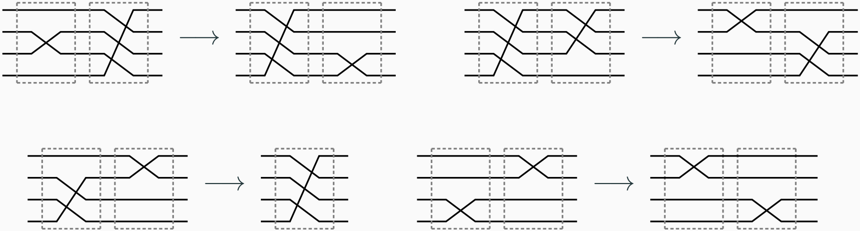
The other conditions $k_i < k_{i+1}$ are satisfied with the following new rewrite rules.



Rewriting system

$$\sigma_{0,n} \longrightarrow \mathrm{id}_n \quad \text{(R11)} \qquad \sigma_{n,0} \longrightarrow \mathrm{id}_n \quad \text{(R12)}$$

$$\sigma_{n,m+1} \longrightarrow (\sigma_{n,m} \otimes \mathrm{id}_1) \mathbin{;} (\mathrm{id}_m \otimes \sigma_{n,1}) \quad \text{(R13)}$$

$$\begin{gathered} L^{k_1}_{\ell_1}(\tau_{d_1}) \mathbin{;} L^{k_2}_{\ell_2}(\tau_{d_2}) \longrightarrow L^{k_2}_{\ell_2}(\tau_{d_2}) \mathbin{;} L^{k_1+1}_{\ell_1-1}(\tau_{d_1}) \\ \text{if } \ d_1 \geq 1, d_2 \geq 1 \ \text{ and } \ k_2 \leq k_1 < k_2 + d_2 - d_1 \end{gathered} \qquad \text{(R14)}$$

$$\begin{gathered} L^{k_1}_{\ell_1}(\tau_{d_1}) \mathbin{;} L^{k_2}_{\ell_2}(\tau_{d_2}) \longrightarrow L^{k_2}_{\ell_2+1}(\tau_{d_2-1}) \mathbin{;} L^{k_1+1}_{\ell_1}(\tau_{d_1-1}) \\ \text{if } \ d_1 \geq 1, d_2 \geq 1 \ \text{ and } \ k_2 \leq k_1 \wedge k_2 + d_2 - d_1 \leq k_1 < k_2 + d_2 \end{gathered} \qquad \text{(R15)}$$

$$\begin{gathered} L^{k_1}_{\ell_1}(\tau_{d_1}) \mathbin{;} L^{k_2}_{\ell_2}(\tau_{d_2}) \longrightarrow L^{k_2}_{\ell_1}(\tau_{d_1+d_2}) \\ \text{if } \ d_1 \geq 1, d_2 \geq 1 \ \text{ and } \ k_1 = k_2 + d_2 \end{gathered} \qquad \text{(R16)}$$

$$\begin{gathered} L^{k_1}_{\ell_1}(\tau_{d_1}) \mathbin{;} L^{k_2}_{\ell_2}(\tau_{d_2}) \longrightarrow L^{k_2}_{\ell_2}(\tau_{d_2}) \mathbin{;} L^{k_1}_{\ell_1}(\tau_{d_1}) \\ \text{if } \ d_1 \geq 1, d_2 \geq 1 \ \text{ and } \ k_1 > k_2 + d_2 \end{gathered} \qquad \text{(R17)}$$

Result

Theorem

The rewriting system is terminating and confluent.

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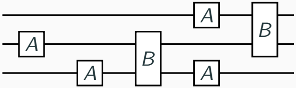
The key steps of the proofs have been verified using the proof assistant Isabelle/HOL.

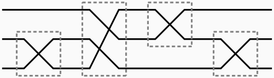
Future work

We solved two sub-problems.

(1) Diagrams without swaps.



(2) Diagrams with only swaps.



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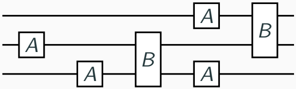

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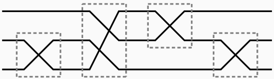
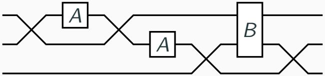


The next step is to combine the two solutions to solve the general problem.

(3) Diagrams with swaps and boxes.



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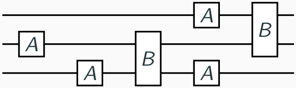

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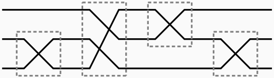

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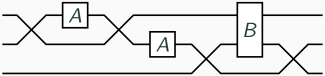

Thanks!