

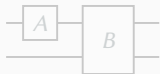
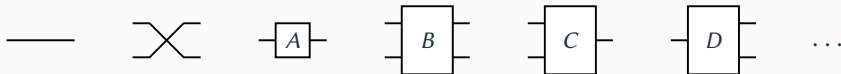
Diagrammatic Reasoning with Control as a Constructor, Applications to Quantum Circuits

14th April 2026 - FoSSaCS 2026

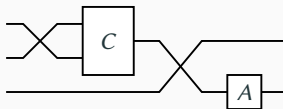
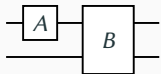
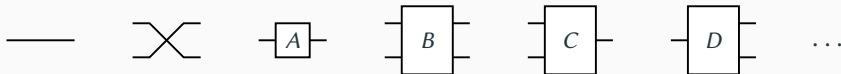
Noé Delorme and Simon Perdrix

Université de Lorraine, CNRS, INRIA, LORIA, Nancy, France

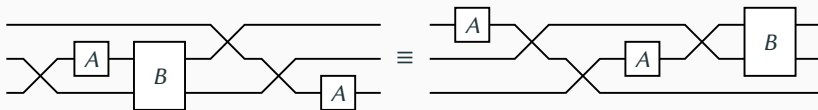
Circuits as generators and compositions



Circuits as generators and compositions

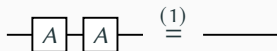


Diagrammatic equivalence



$\equiv$: diagrammatic equivalence

Circuit rewriting



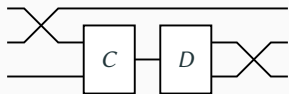
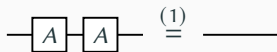
(1)



(2)



Circuit rewriting



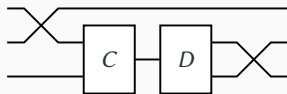
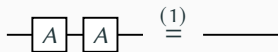
$\stackrel{(1)}{=}$



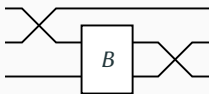
$\stackrel{(2)}{=}$



Circuit rewriting



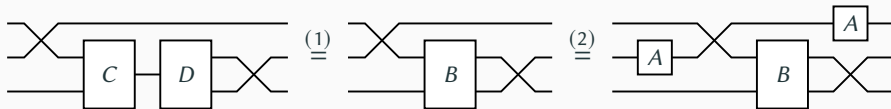
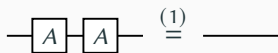
$\stackrel{(1)}{=}$



$\stackrel{(2)}{=}$

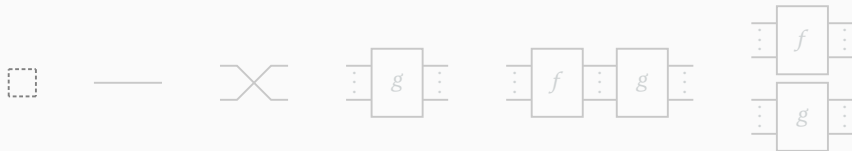


Circuit rewriting



PROP formalism

$$\begin{array}{c}
 \overline{\text{id}_0 : 0 \rightarrow 0} \quad \overline{\text{id}_1 : 1 \rightarrow 1} \quad \overline{\sigma : 2 \rightarrow 2} \quad \overline{g : \text{dom}(g) \rightarrow \text{cod}(g)} \\
 \text{where } g \in \Sigma \\
 \\
 \frac{f : n \rightarrow p \quad g : p \rightarrow m}{f \circ g : n \rightarrow m} \quad \frac{f : n \rightarrow m \quad g : p \rightarrow q}{f \otimes g : n + p \rightarrow m + q}
 \end{array}$$



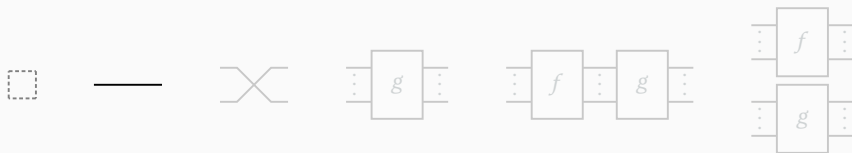
Σ : a set of generators

$\circ$: the sequential composition

$\otimes$: the parallel composition

PROP formalism

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 \overline{\text{id}_0 : 0 \rightarrow 0} \quad \overline{\text{id}_1 : 1 \rightarrow 1} \quad \overline{\sigma : 2 \rightarrow 2} \quad \frac{g \in \Sigma}{\overline{g : \text{dom}(g) \rightarrow \text{cod}(g)}} \\
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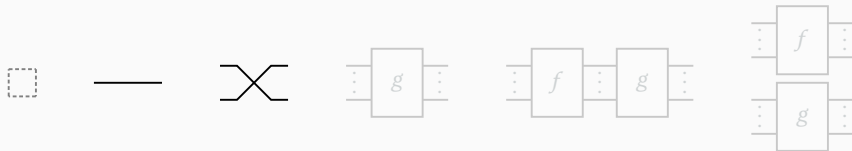
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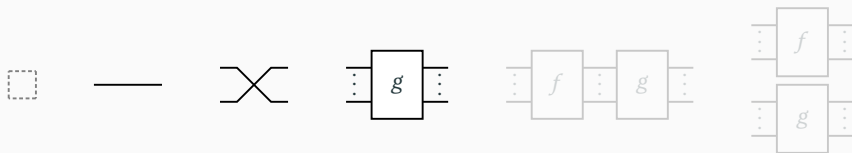
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 \end{array}$$



Σ : a set of generators

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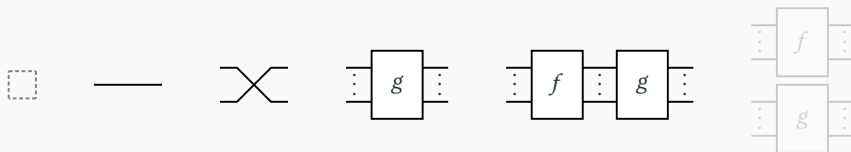
$\otimes$: the parallel composition

PROP formalism

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$$\frac{f : n \rightarrow m \quad g : p \rightarrow q}{f \otimes g : n + p \rightarrow m + q}$$



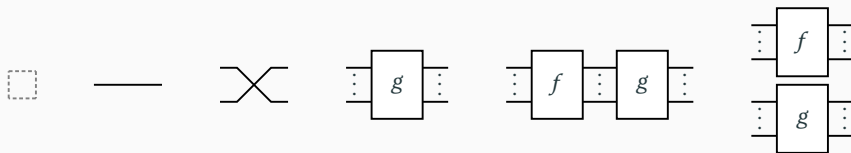
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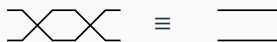


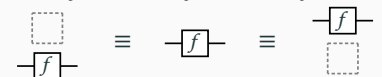
Σ : a set of generators

$\circ$: the sequential composition

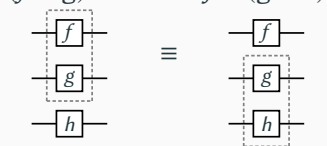
$\otimes$: the parallel composition

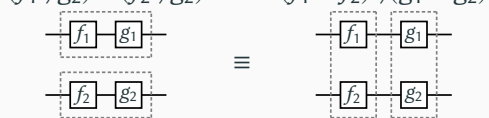
PROP formalism

$$\sigma \circ \sigma \equiv \text{id}_1 \otimes \text{id}_1$$


$$\text{id}_0 \otimes f \equiv f \equiv f \otimes \text{id}_0$$


$$(f \circ g) \circ h \equiv f \circ (g \circ h)$$


$$(f \otimes g) \otimes h \equiv f \otimes (g \otimes h)$$


$$(f_1 \circ g_2) \otimes (f_2 \circ g_2) \equiv (f_1 \otimes f_2) \circ (g_1 \otimes g_2)$$


...

PROP formalism

$$\frac{f \equiv g}{\mathbf{P}/\mathcal{R} \vdash f = g}$$

$$\frac{(L = R) \in \mathcal{R}}{\mathbf{P}/\mathcal{R} \vdash L = R}$$

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash h \circ f = h \circ g}$$

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash f \circ h = g \circ h}$$

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash h \otimes f = h \otimes g}$$

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$\equiv$: diagrammatic equivalence

$\mathbf{P}$: a PROP

$\mathcal{R}$: a set of rewrite equations

$\mathbf{P}/\mathcal{R} \vdash \cdot = \cdot$: the equivalence relation

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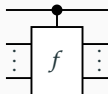
$\equiv$: diagrammatic equivalence

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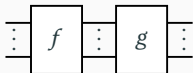
Standard control



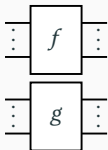
 : initialization of state 0
 : initialization of state 1

Controlled PROP

$$\frac{f : n \rightarrow p \quad g : p \rightarrow m}{f \circ g : n \rightarrow m}$$

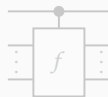


$$\frac{f : n \rightarrow m \quad g : p \rightarrow q}{f \otimes g : n + p \rightarrow m + q}$$



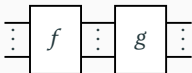
NEW!

$$\frac{f : n \rightarrow n}{C(f) : 1 + n \rightarrow 1 + n}$$

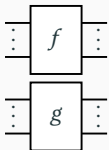


Controlled PROP

$$\frac{f : n \rightarrow p \quad g : p \rightarrow m}{f \circ g : n \rightarrow m}$$

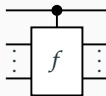


$$\frac{f : n \rightarrow m \quad g : p \rightarrow q}{f \otimes g : n + p \rightarrow m + q}$$



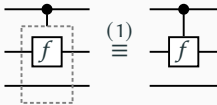
NEW!

$$\frac{f : n \rightarrow n}{C(f) : 1 + n \rightarrow 1 + n}$$



Controlled PROP

NEW!



NEW!

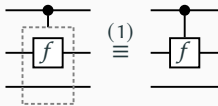


NEW!

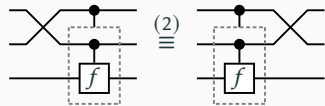


Controlled PROP

NEW!



NEW!

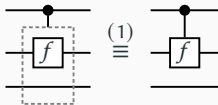


NEW!

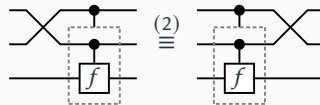


Controlled PROP

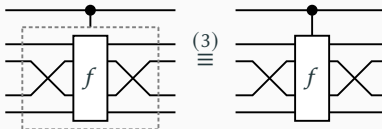
NEW!



NEW!



NEW!



Controlled PROP

Definition

A **control functor** maps any endocircuit $f : n \rightarrow n$ to the endocircuit $C(f) : (1 + n) \rightarrow (1 + n)$ while satisfying (1), (2), and (3).

Definition

A **controlled PROP** is a PROP equipped with a control functor.

Controlled PROP

Definition

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Definition

A **controlled PROP** is a PROP equipped with a control functor.

Controlled PROP

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

$$C(f \circ g) = C(f) \circ C(g)$$

Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Controlled PROP

$$C(\text{id}_1) = \text{id}_1 \otimes \text{id}_1$$

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Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

(3)

(1)

Controlled PROP

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(3) $C(f \otimes g) = C(f) \otimes C(g)$

(1) $C(f \otimes g) = C(f) \otimes C(g)$

Controlled PROP

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(3)

(1)

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(3)

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$$C(f \circ g) = C(f) \circ C(g)$$

Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

Diagrammatic equation showing the relationship between $C(f \otimes g)$ and $C(f) \otimes C(g)$. The equation is composed of several parts connected by equals signs:

- Leftmost part: A control line with a dot connected to a dashed box containing two parallel wires, each with a box labeled f and g respectively.
- Second part: A control line with a dot connected to a dashed box containing two crossing wires, with boxes f and g placed on the wires before the crossing.
- Third part: A control line with a dot connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.
- Fourth part: A control line with two dots connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.
- Fifth part: A control line with two dots connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.
- Sixth part: A control line with two dots connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.
- Seventh part: A control line with two dots connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.
- Eighth part: A control line with two dots connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.
- Ninth part: A control line with two dots connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.
- Tenth part: A control line with two dots connected to two separate dashed boxes, each containing a crossing wire with a box f and g respectively.

Controlled PROP

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash C(f) = C(g)} \quad \text{NEW!}$$

Example

Let **CNot** be the controlled PROP generated by $\text{---}\oplus\text{---}$ with the following rewrite equations.

$$\text{---}\oplus\text{---}\oplus\text{---} = \text{---}, \quad \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array} = \begin{array}{c} \text{---} \\ \oplus \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array}, \quad \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} = \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array}.$$

Then, $\begin{array}{c} \bullet \\ \text{---} \\ \oplus \end{array}$ is not a generator. It is the circuit $C(\text{---}\oplus\text{---})$.

Moreover, the equations $\begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \oplus \quad \oplus \end{array} = \text{---} \quad \text{---}$ is derivable.

Controlled PROP

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash C(f) = C(g)} \quad \text{NEW!}$$

Example

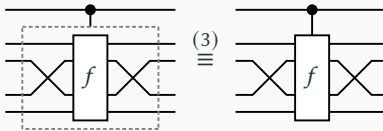
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$$\text{---}\oplus\text{---}\oplus\text{---} = \text{---}, \quad \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array} = \begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array}, \quad \begin{array}{c} \oplus \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array} = \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \oplus \end{array} \begin{array}{c} \oplus \\ \text{---} \end{array}.$$

Then, $\begin{array}{c} \bullet \\ \text{---} \\ \oplus \end{array}$ is not a generator. It is the circuit $C(\text{---}\oplus\text{---})$.

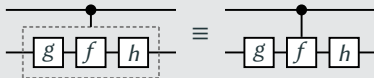
Moreover, the equations $\begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \oplus \quad \oplus \end{array} = \text{---} \quad \text{---}$ is derivable.

Conjugated PROP



Definition

A **conjugated PROP** is a controlled PROP such that



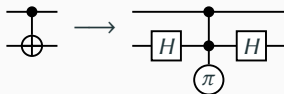
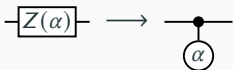
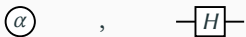
is satisfied whenever $\mathbf{P}/\mathcal{R} \vdash \boxed{g} \boxed{h} = \text{---}$.

Applications to Quantum Circuits

Quantum circuits



Quantum circuits



Quantum circuits

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$\boxed{Z(2\pi)} = \text{---}$$

$$\boxed{Z(\alpha_1)} \boxed{Z(\alpha_2)} = \boxed{Z(\alpha_1 + \alpha_2)}$$

$$\boxed{X(\alpha_1)} \boxed{H} \boxed{X(\alpha_2)} = \textcircled{\beta_0} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)}$$

$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\alpha)} \begin{array}{c} \bullet \\ \oplus \end{array} = \boxed{Z(\alpha)}$$

$$\begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$$

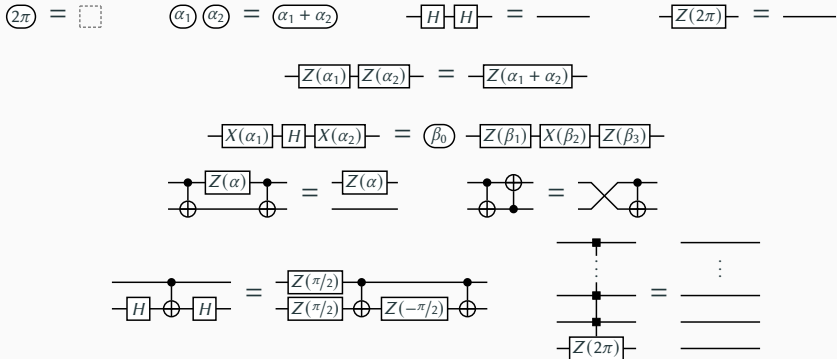
$$\begin{array}{c} \bullet \\ \oplus \end{array} \boxed{H} \begin{array}{c} \bullet \\ \oplus \end{array} = \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array} \boxed{Z(-\pi/2)} \begin{array}{c} \bullet \\ \oplus \end{array}$$

$$\begin{array}{c} \blacksquare \\ \vdots \\ \blacksquare \\ \blacksquare \\ \boxed{Z(2\pi)} \end{array} = \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

Theorem

This **minimal** set of rewrite equations is **complete**.

Quantum circuits



Theorem

This **minimal** set of rewrite equations is **complete**.

Quantum circuits

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \textcircled{2\pi} = \text{---}$$

$$\begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \textcircled{\alpha_1} \quad \textcircled{\alpha_2} \end{array} = \begin{array}{c} \bullet \\ | \\ \textcircled{\alpha_1 + \alpha_2} \end{array}$$

$$\text{---} \boxed{H} \textcircled{\alpha_1} \text{---} \boxed{H} \textcircled{\alpha_2} \text{---} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \textcircled{\beta_1} \boxed{H} \textcircled{\beta_2} \boxed{H} \textcircled{\beta_3} \text{---}$$

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ \textcircled{\pi} \quad \textcircled{\alpha} \quad \textcircled{\pi} \end{array} = \begin{array}{c} \bullet \\ | \\ \textcircled{\alpha} \end{array}$$

$$\text{X} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ \boxed{H} \quad \boxed{H} \quad \boxed{H} \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \textcircled{\pi} \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \quad | \quad | \quad | \\ \textcircled{\pi/2} \quad \textcircled{\pi/2} \quad \boxed{H} \quad \textcircled{\pi} \quad \boxed{H} \quad \textcircled{-\pi/2} \quad \boxed{H} \quad \textcircled{\pi} \quad \boxed{H} \end{array}$$

$$\begin{array}{c} \bullet \\ \vdots \\ \bullet \\ | \\ \textcircled{2\pi} \end{array} = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}$$

Quantum circuits

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \textcircled{2\pi} \text{---} = \text{---}$$

$$\begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \textcircled{\alpha_1} \quad \textcircled{\alpha_2} \end{array} = \begin{array}{c} \bullet \\ | \\ \textcircled{\alpha_1 + \alpha_2} \end{array}$$

$$\text{---} \boxed{H} \bullet \boxed{H} \bullet \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \bullet \boxed{H} \bullet \boxed{H} \bullet \text{---}$$

$$\textcircled{\alpha_1} \quad \textcircled{\alpha_2} \quad \quad \quad \textcircled{\beta_1} \quad \textcircled{\beta_2} \quad \textcircled{\beta_3}$$

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ \textcircled{\pi} \quad \textcircled{\alpha} \quad \textcircled{\pi} \end{array} = \begin{array}{c} \bullet \\ | \\ \textcircled{\alpha} \end{array}$$

$$\text{X} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ \boxed{H} \quad \boxed{H} \quad \boxed{H} \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ \textcircled{\pi} \quad \textcircled{\pi} \quad \textcircled{\pi} \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \textcircled{\pi} \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \quad | \quad | \quad | \\ \textcircled{\pi/2} \quad \textcircled{\pi/2} \quad \boxed{H} \quad \textcircled{\pi} \quad \boxed{H} \quad \textcircled{-\pi/2} \quad \boxed{H} \quad \textcircled{\pi} \quad \boxed{H} \end{array}$$

$$\begin{array}{c} \bullet \\ \vdots \\ \bullet \\ | \\ \textcircled{2\pi} \end{array} = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}$$

Quantum circuits

$$\textcircled{2\pi} = \square$$

$$\alpha_1 \alpha_2 = \alpha_1 + \alpha_2$$

$$\text{---} \boxed{H} \text{---} \boxed{H} \text{---} = \text{---}$$

$$\text{---}\bullet\text{---} = \text{---}$$

$$\begin{array}{c} \text{---}[H]\text{---} \\ | \\ \text{---} \end{array} \bullet \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \text{---} \end{array} \bullet \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \text{---} \end{array} = (\beta_0) \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \end{array} [H] \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \end{array} [H] \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \end{array}$$

The diagram shows a vertical stack of four horizontal lines. The bottom line is enclosed in a circle with the label 2π . A vertical line with dots connects the top and bottom lines. This is followed by an equals sign and a single horizontal line with a vertical line and dots above it, representing a phase shift.

Quantum circuits

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \textcircled{2\pi} \text{---} = \text{---}$$

$$\text{---} \textcircled{\alpha_1} \textcircled{\alpha_2} \text{---} = \text{---} \textcircled{\alpha_1 + \alpha_2} \text{---}$$

$$\text{---} \boxed{H} \textcircled{\alpha_1} \boxed{H} \textcircled{\alpha_2} \boxed{H} \text{---} = \textcircled{\beta_0} \textcircled{\beta_1} \boxed{H} \textcircled{\beta_2} \boxed{H} \textcircled{\beta_3} \text{---}$$

$$\text{---} \textcircled{\pi} \textcircled{\alpha} \textcircled{\pi} \text{---} = \text{---} \textcircled{\alpha} \text{---}$$

$$\text{---} \text{---} = \text{---} \boxed{H} \textcircled{\pi} \boxed{H} \textcircled{\pi} \boxed{H} \textcircled{\pi} \boxed{H} \text{---}$$

$$\text{---} \textcircled{\pi} \text{---} = \text{---} \textcircled{\pi/2} \textcircled{\pi/2} \boxed{H} \textcircled{\pi} \boxed{H} \textcircled{-\pi/2} \textcircled{\pi} \boxed{H} \text{---}$$

$$\text{---} \textcircled{2\pi} \text{---} = \text{---}$$

Quantum circuits

$$\textcircled{2\pi} = \square$$

$$\alpha_1 \alpha_2 = \alpha_1 + \alpha_2$$

$$\boxed{H} \boxed{H} = \text{---}$$

$$\text{---}\bullet\text{---} = \text{---}$$

$$\begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\alpha_1} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\alpha_2} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\alpha_3} \end{array} = \textcircled{\beta_0} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\beta_1} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\beta_2} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\beta_3} \end{array}$$

The diagram illustrates the decomposition of a CNOT gate into a sequence of Hadamard (H) and Toffoli gates. On the left, a CNOT gate is shown with a control line (top) and a target line (bottom). This is equated to a sequence of gates on the right. The sequence starts with a Hadamard (H) gate on the target line, followed by a Toffoli gate with the control line as control and the target line as target. This is followed by another H gate on the target line, then a Toffoli gate with the control line as control and the target line as target. This sequence is repeated three times, resulting in a total of six H gates on the target line and three Toffoli gates. The final output is the target line, which is the result of the CNOT operation.

The diagram illustrates the equivalence between two representations of a multi-qubit system. On the left, a vertical stack of lines represents a system with a phase shift of 2π applied to the bottom line. On the right, a horizontal stack of lines represents the same system without the phase shift. The two representations are shown to be equivalent with an equals sign between them.

Quantum circuits

$$\textcircled{2\pi} = \square$$

$$\alpha_1 \alpha_2 = \alpha_1 + \alpha_2$$

$$\text{---} \boxed{H} \text{---} \boxed{H} \text{---} = \text{---}$$

$$\text{---}\bullet\text{---} = \text{---}$$

$$\begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\alpha_1} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\alpha_2} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\alpha_3} \end{array} = \textcircled{\beta_0} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\beta_1} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\beta_2} \end{array} \begin{array}{c} \text{---}[H]\text{---} \\ | \\ \textcircled{\beta_3} \end{array}$$

The diagram illustrates the decomposition of a CNOT gate into a sequence of Hadamard (H) and Toffoli gates. On the left, a CNOT gate is shown with a control line (top) and a target line (bottom). This is equated to a sequence of gates on the right. The sequence starts with a Hadamard (H) gate on the target line, followed by a Toffoli gate with the control line as the first control and the target line as the second control. This is followed by another H gate on the target line, then another Toffoli gate with the control line as the first control and the target line as the second control. Finally, there is a third H gate on the target line. The target line is also marked with a π phase shift at the end of the sequence.

Quantum circuits

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \textcircled{2\pi} \text{---} = \text{---}$$

$$\text{---} \textcircled{\alpha_1} \textcircled{\alpha_2} \text{---} = \text{---} \textcircled{\alpha_1 + \alpha_2} \text{---}$$

$$\text{---} \boxed{H} \bullet \boxed{H} \bullet \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \bullet \boxed{H} \bullet \boxed{H} \bullet \text{---}$$

$\textcircled{\alpha_1} \quad \textcircled{\alpha_2} \qquad \qquad \textcircled{\beta_1} \quad \textcircled{\beta_2} \quad \textcircled{\beta_3}$

$$\text{---} \textcircled{\pi} \textcircled{\alpha} \textcircled{\pi} \text{---} = \text{---} \textcircled{\alpha} \text{---}$$

$$\text{X} = \text{---} \boxed{H} \bullet \boxed{H} \bullet \boxed{H} \bullet \text{---}$$

$\textcircled{\pi} \quad \textcircled{\pi} \quad \textcircled{\pi}$

$$\text{---} \textcircled{\pi} \text{---} = \text{---} \textcircled{\pi/2} \textcircled{\pi/2} \boxed{H} \textcircled{\pi} \textcircled{-\pi/2} \textcircled{\pi} \text{---}$$

$$\text{---} \vdots \text{---} = \text{---} \vdots \text{---}$$

$\textcircled{2\pi}$

Quantum circuits

$$\textcircled{2\pi} = \boxed{}$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \textcircled{2\pi} \text{---} = \text{---}$$

$$\text{---} \textcircled{\alpha_1} \textcircled{\alpha_2} \text{---} = \text{---} \textcircled{\alpha_1 + \alpha_2} \text{---}$$

$$\text{---} \boxed{H} \textcircled{\alpha_1} \boxed{H} \textcircled{\alpha_2} \boxed{H} \text{---} = \textcircled{\beta_0} \text{---} \textcircled{\beta_1} \boxed{H} \textcircled{\beta_2} \boxed{H} \textcircled{\beta_3} \text{---}$$

$$\text{---} \textcircled{\pi} \textcircled{\alpha} \textcircled{\pi} \text{---} = \text{---} \textcircled{\alpha} \text{---}$$

$$\text{---} \text{---} = \text{---} \boxed{H} \textcircled{\pi} \boxed{H} \textcircled{\pi} \boxed{H} \textcircled{\pi} \boxed{H} \text{---}$$

$$\text{---} \textcircled{\pi} \text{---} = \text{---} \textcircled{\pi/2} \textcircled{\pi/2} \boxed{H} \textcircled{\pi} \boxed{H} \textcircled{-\pi/2} \textcircled{\pi} \boxed{H} \text{---}$$

$$\text{---} \textcircled{2\pi} \text{---} = \text{---}$$

Quantum circuits

$$(2\pi) = \square$$

$$(\alpha_1)(\alpha_2) = (\alpha_1 + \alpha_2)$$

$$-H-H- = -$$

$$H \cdot H \cdot H = (\beta_0) \cdot H \cdot H$$

Diagram showing a sequence of three Hadamard gates with phase shifts α_1 and α_2 on the control lines, equated to a single Hadamard gate with phase shift β_0 and two Hadamard gates with phase shifts β_1 and β_2 on the control lines.

$$\text{X} = H \cdot H \cdot H$$

Diagram showing a CNOT gate (X) with control and target lines, equated to a sequence of three Hadamard gates with phase shifts π on the control lines.

Quantum circuits

$$\textcircled{2\pi} = \square$$

$$\textcircled{\alpha_1} \textcircled{\alpha_2} = \textcircled{\alpha_1 + \alpha_2}$$

$$\text{---} \boxed{H} \boxed{H} \text{---} = \text{---}$$

$$\text{---} \boxed{X(\alpha_1)} \boxed{H} \boxed{X(\alpha_2)} \text{---} = \textcircled{\beta_0} \text{---} \boxed{Z(\beta_1)} \boxed{X(\beta_2)} \boxed{Z(\beta_3)} \text{---}$$

$$\text{X} = \begin{array}{c} \bullet \quad \oplus \quad \bullet \\ | \quad | \quad | \\ \oplus \quad \bullet \quad \oplus \end{array}$$

To go beyond

Polycontrolled PROPs.

Applications to reversible classical circuits.

Compilation of quantum circuits.

Thanks



(arXiv:2508.21756)

**Diagrammatic Reasoning with Control as a Constructor,
Applications to Quantum Circuits**