Diagrammatic Reasoning with Control as a Constructor, Applications to Quantum Circuits

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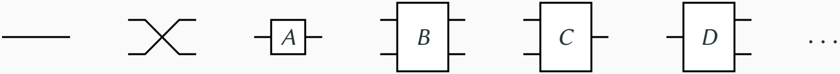
Circuits as generators and compositions



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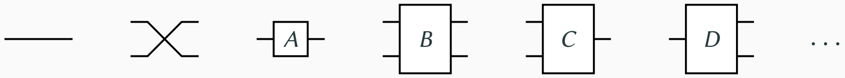
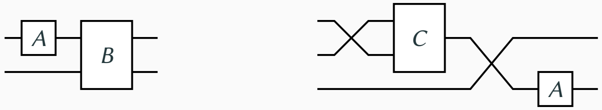
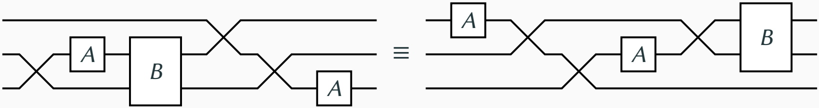


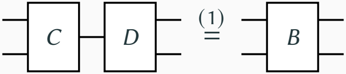


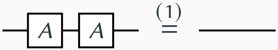
Diagrammatic equivalence



$\equiv$: diagrammatic equivalence

Circuit rewriting





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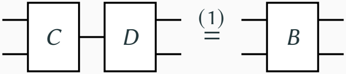
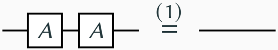
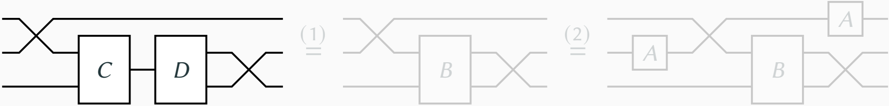
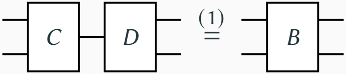
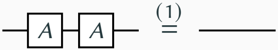
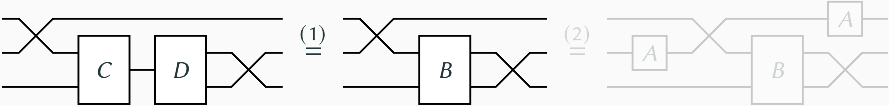


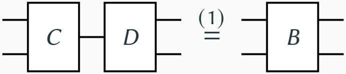





C
D
(1)
B


A
A
(1)


C
D
(1)
B
(2)
A
A
B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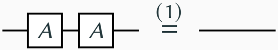
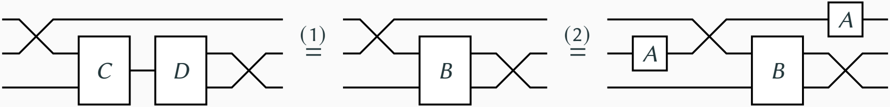
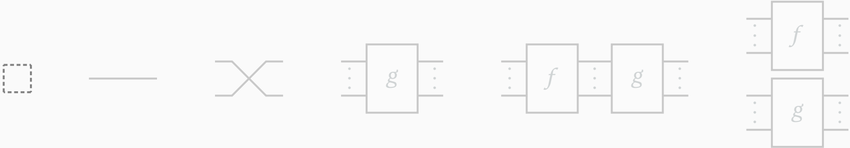

A
A
(1)


C
D
(1)
B
(2)
A
B
A

PROP formalism

$$\frac{}{\mathrm{id}_0 : 0 \to 0} \qquad \frac{}{\mathrm{id}_1 : 1 \to 1} \qquad \frac{}{\sigma : 2 \to 2} \qquad \frac{g \in \Sigma}{g : \mathrm{dom}(g) \to \mathrm{cod}(g)}$$

$$\frac{f : n \to p \quad g : p \to m}{f \mathbin{;} g : n \to m} \qquad \frac{f : n \to m \quad g : p \to q}{f \otimes g : n + p \to m + q}$$



Σ : a set of generators

$;$: the sequential composition

$\otimes$: the parallel composition

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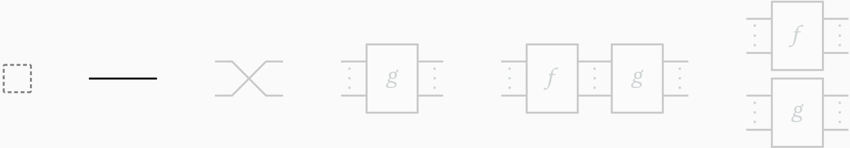

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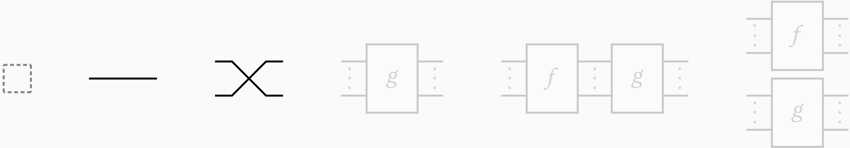

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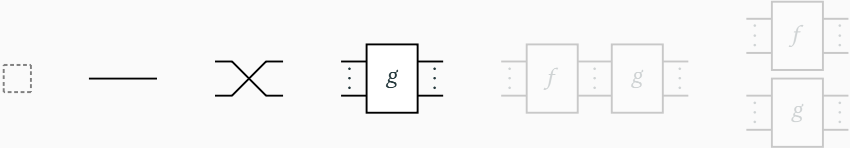

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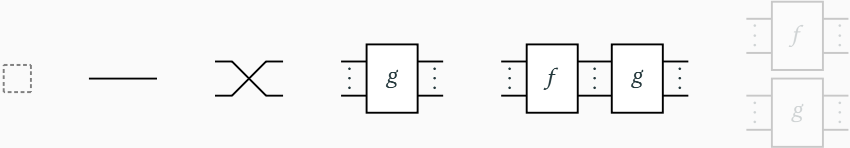

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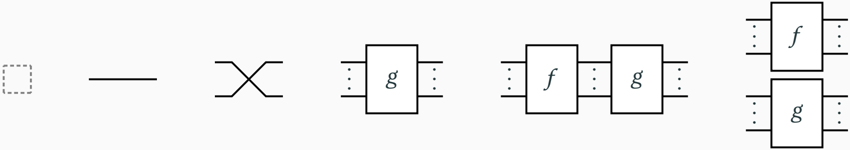

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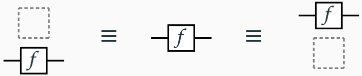
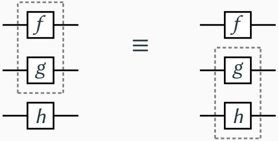
$$\sigma \fatsemi \sigma \equiv \mathrm{id}_1 \otimes \mathrm{id}_1$$

$$\mathrm{id}_0 \otimes f \equiv f \equiv f \otimes \mathrm{id}_0$$

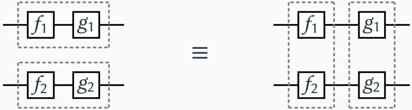


$$(f \fatsemi g) \fatsemi h \equiv f \fatsemi (g \fatsemi g)$$

$$(f \otimes g) \otimes h \equiv f \otimes (g \otimes h)$$



$$(f_1 \fatsemi g_2) \otimes (f_2 \fatsemi g_2) \equiv (f_1 \otimes f_2) \fatsemi (g_1 \otimes g_2)$$



$\ldots$

PROP formalism

$$\frac{f \equiv g}{\mathbf{P}/\mathcal{R} \vdash f = g} \qquad \frac{(L = R) \in \mathcal{R}}{\mathbf{P}/\mathcal{R} \vdash L = R}$$

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash h \fatsemi f = h \fatsemi g} \qquad \frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash f \fatsemi h = g \fatsemi h}$$

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$\mathbf{P}$: a PROP

$\mathcal{R}$: a set of rewrite equations

$\mathbf{P}/\mathcal{R} \vdash \cdot = \cdot$: the equivalence relation

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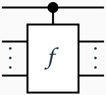
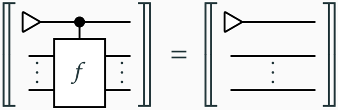
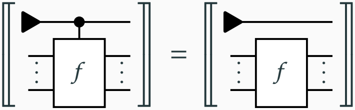




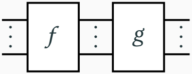


▷— : initialization of state 0

▶— : initialization of state 1

Controlled PROP

$$\frac{f : n \to p \quad g : p \to m}{f \fatsemi g : n \to m}$$



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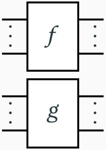
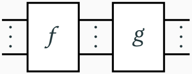


NEW!

$$\frac{f : n \to n}{\mathrm{C}(f) : 1 + n \to 1 + n}$$

Controlled PROP

$$\frac{f : n \to p \quad g : p \to m}{f \mathbin{;} g : n \to m}$$



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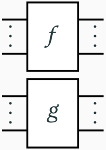

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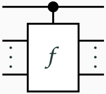
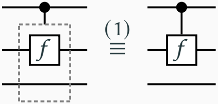


Controlled PROP

NEW!



NEW!

NEW!

Controlled PROP

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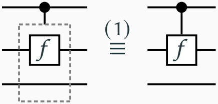

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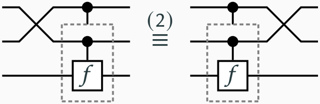

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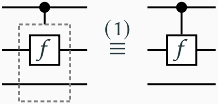

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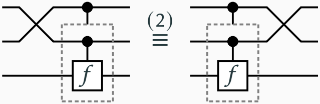

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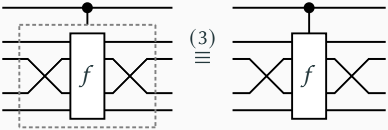

Controlled PROP

Definition

A control functor maps any endocircuit $f : n \to n$ to the endocircuit $C(f) : (1+n) \to (1+n)$ while satisfying (1), (2), and (3).

Definition

A controlled PROP is a PROP equipped with a control functor.

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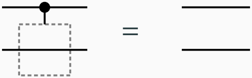
A controlled PROP is a PROP equipped with a control functor.

$$C(\mathrm{id}_1) = \mathrm{id}_1 \otimes \mathrm{id}_1 \qquad C(f \mathbin{;} g) = C(f) \mathbin{;} C(g)$$



Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

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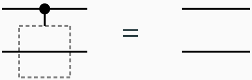

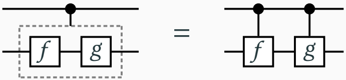

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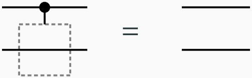
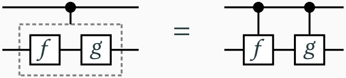
$\mathrm{C}(\mathrm{id}_1) = \mathrm{id}_1 \otimes \mathrm{id}_1$ $\qquad$ $\mathrm{C}(f \fatsemi g) = \mathrm{C}(f) \fatsemi \mathrm{C}(g)$





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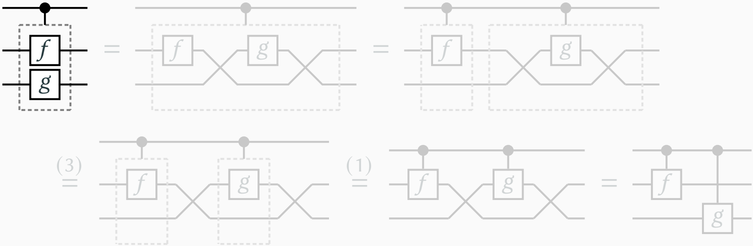

Controlled PROP

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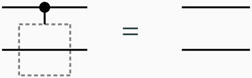

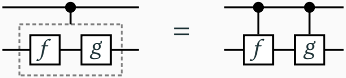

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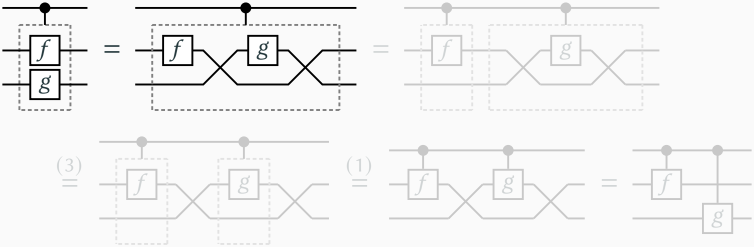

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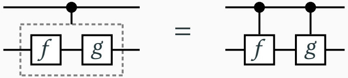

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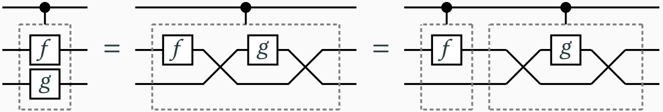
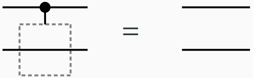
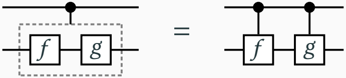


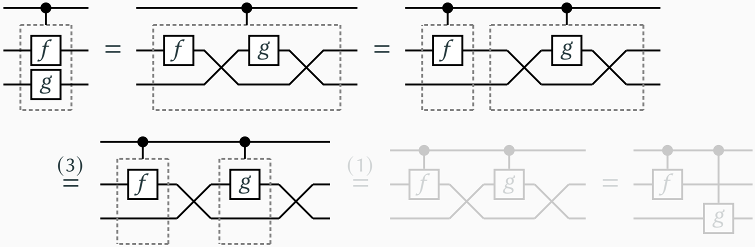
$$\mathrm{C}(\mathrm{id}_1) \quad = \quad \mathrm{id}_1 \otimes \mathrm{id}_1$$



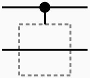
$$\mathrm{C}(f \fatsemi g) \quad = \quad \mathrm{C}(f) \fatsemi \mathrm{C}(g)$$

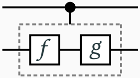
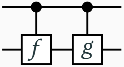


Notice however that $\mathrm{C}(f \otimes g) \neq \mathrm{C}(f) \otimes \mathrm{C}(g)$, instead we have

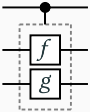
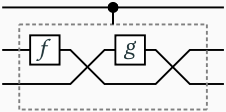
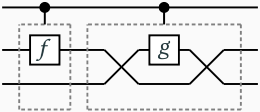


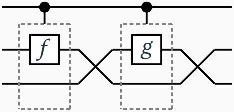
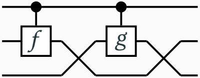
$$C(\mathrm{id}_1) = \mathrm{id}_1 \otimes \mathrm{id}_1 \qquad C(f \mathbin{;} g) = C(f) \mathbin{;} C(g)$$

 = 

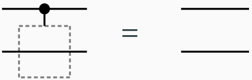
 = 

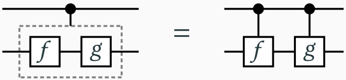
Notice however that $C(f \otimes g) \neq C(f) \otimes C(g)$, instead we have

 =  = 

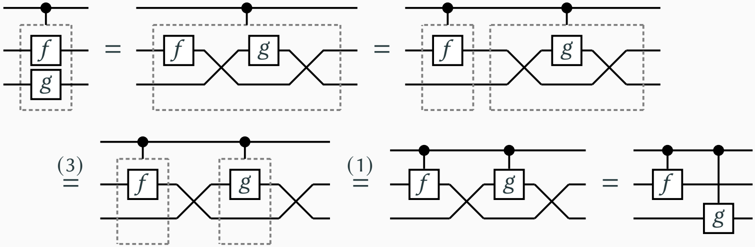
$\overset{(3)}{=}$  $\overset{(1)}{=}$  =

$$\mathrm{C}(\mathrm{id}_1) \quad = \quad \mathrm{id}_1 \otimes \mathrm{id}_1 \qquad\qquad \mathrm{C}(f \fatsemi g) \quad = \quad \mathrm{C}(f) \fatsemi \mathrm{C}(g)$$





Notice however that $\mathrm{C}(f \otimes g) \neq \mathrm{C}(f) \otimes \mathrm{C}(g)$, instead we have



Controlled PROP

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash \mathrm{C}(f) = \mathrm{C}(g)}$$ **NEW!**

Example

Let **CNot** be the controlled PROP generated by with the following rewrite equations.

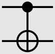
Then, is not a generator. It is the circuit C( ).

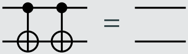
Moreover, the equations is derivable.

$$\frac{\mathbf{P}/\mathcal{R} \vdash f = g}{\mathbf{P}/\mathcal{R} \vdash \mathrm{C}(f) = \mathrm{C}(g)} \quad \textbf{NEW!}$$

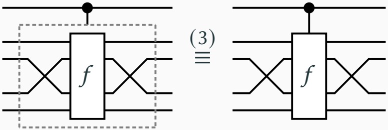
Example

Let **CNot** be the controlled PROP generated by  with the following rewrite equations.

Then,  is not a generator. It is the circuit C().

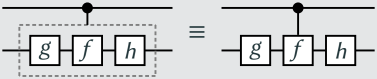
Moreover, the equations  is derivable.

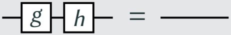
Conjugated PROP



Definition

A conjugated PROP is a controlled PROP such that

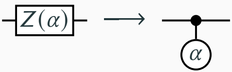


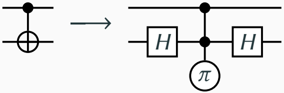
is satisfied whenever $\mathbf{P}/\mathcal{R} \vdash$  .

Applications to Quantum Circuits

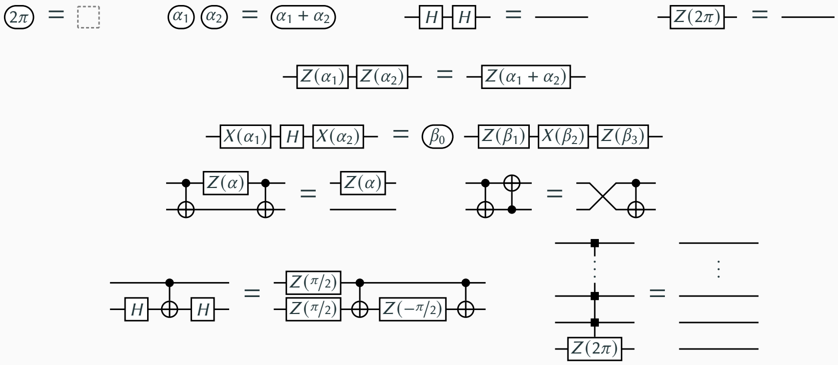
Quantum circuits

Quantum circuits





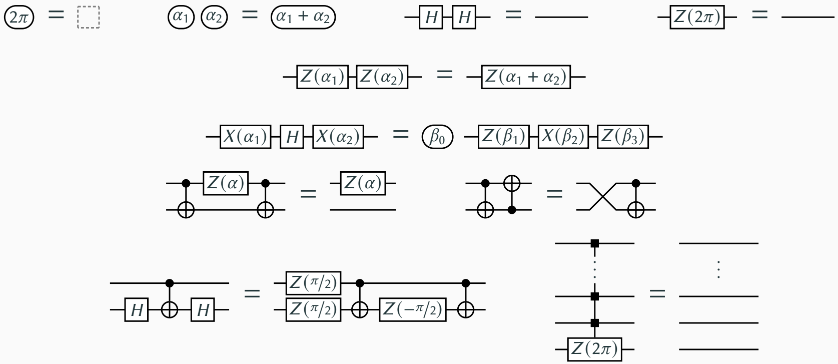
Quantum circuits



Theorem

This minimal set of rewrite equations is complete.

Quantum circuits



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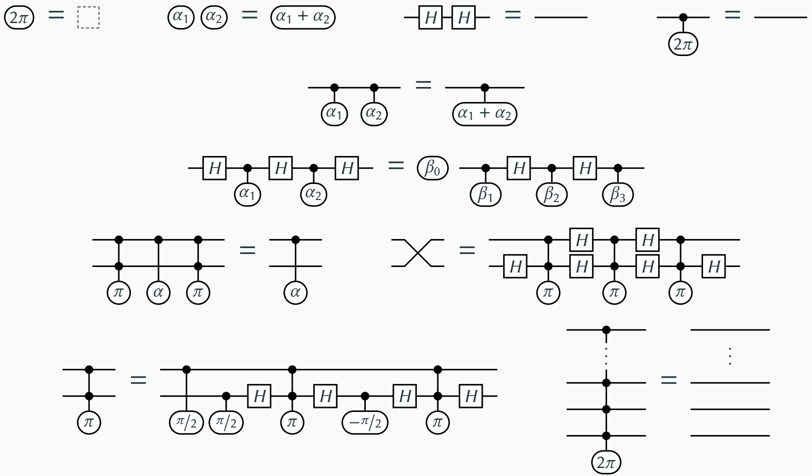
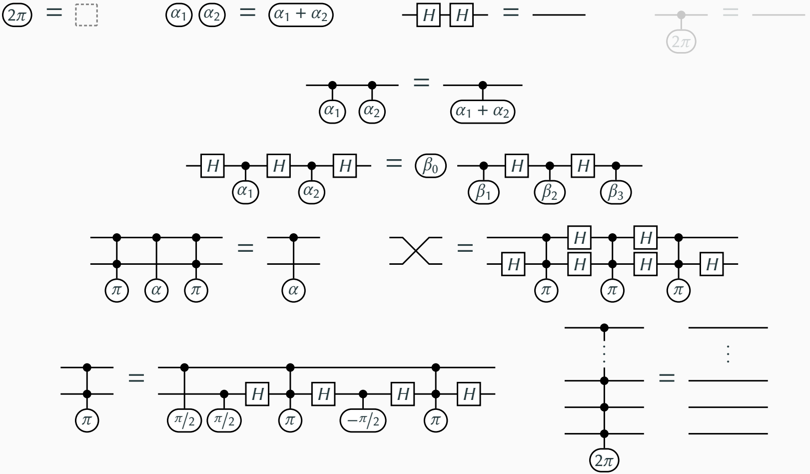
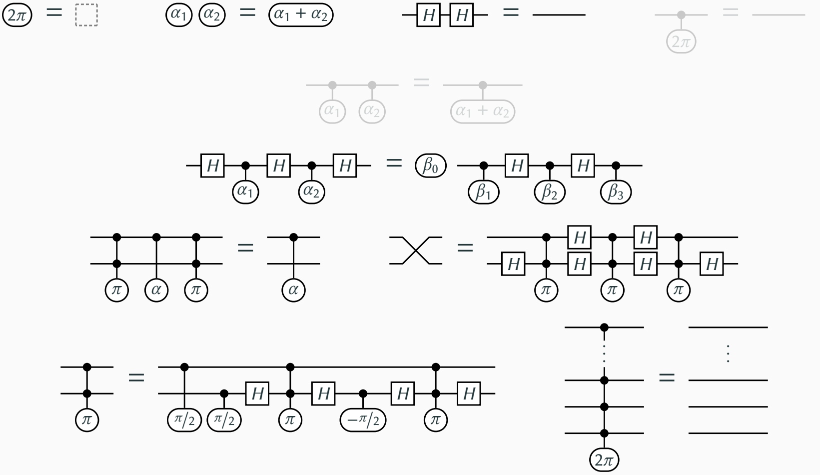


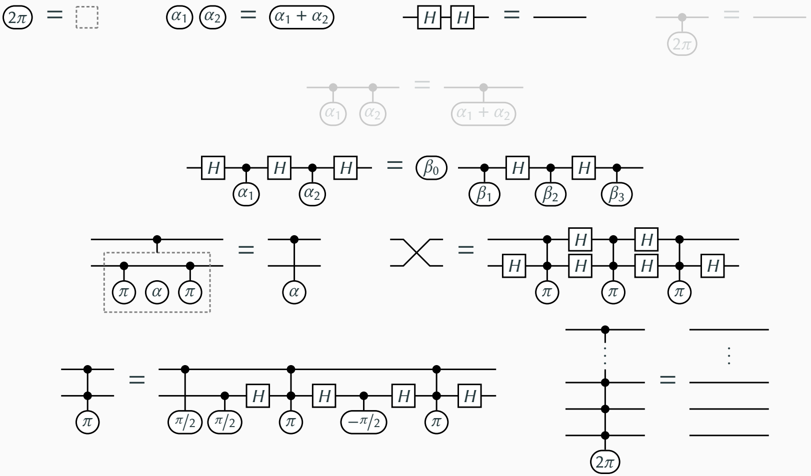
Quantum circuits



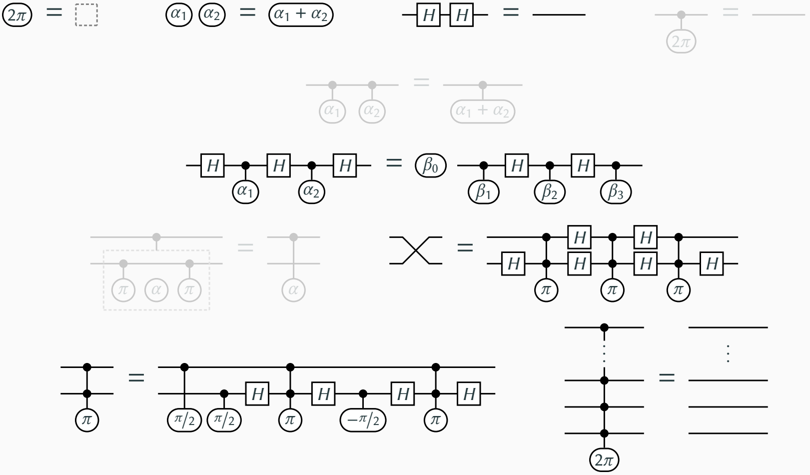
Quantum circuits



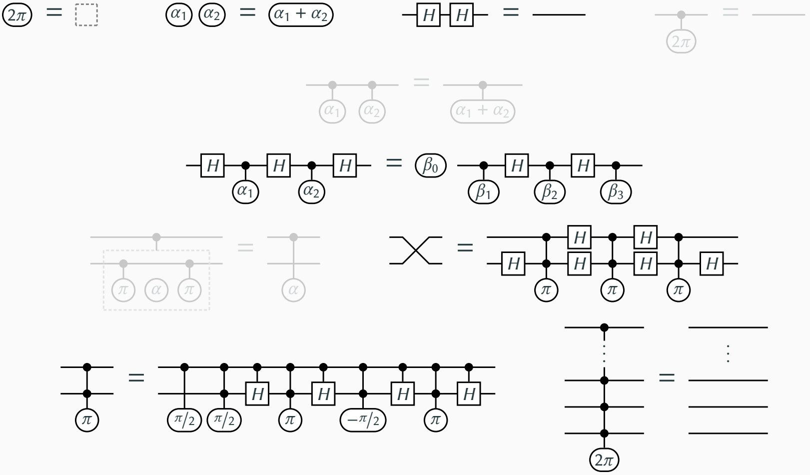
Quantum circuits



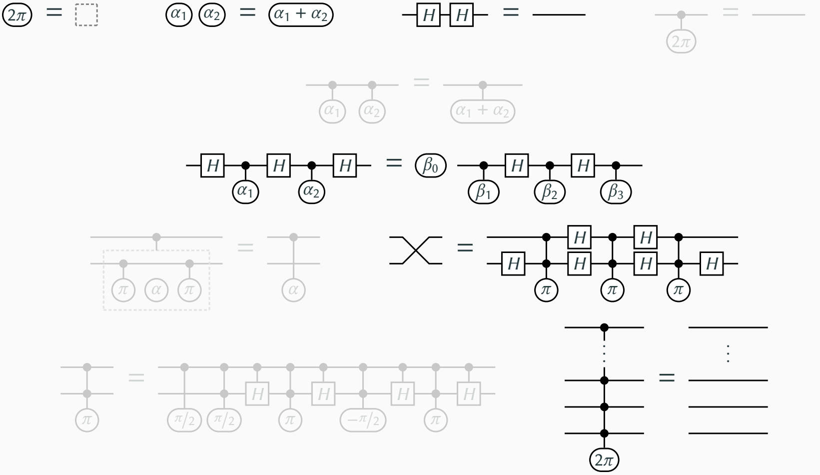
Quantum circuits

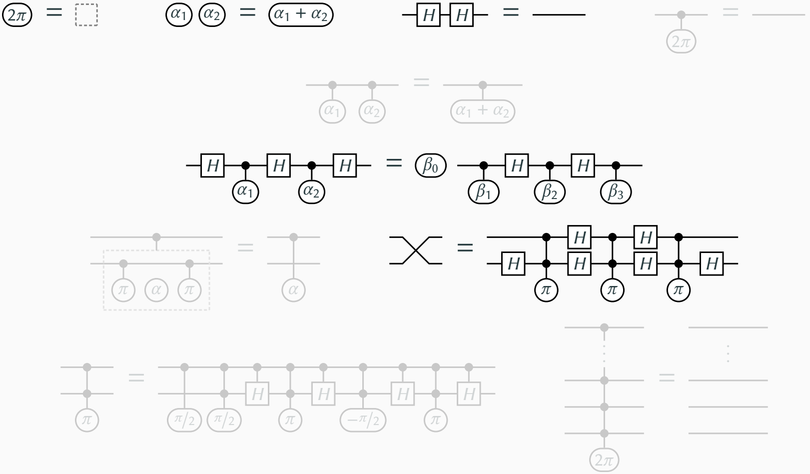


Quantum circuits

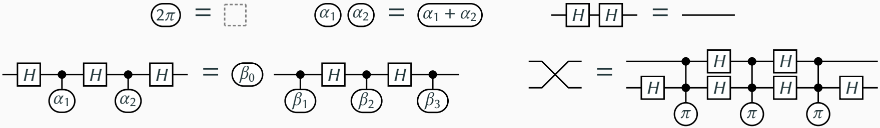


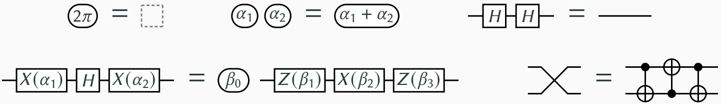
Quantum circuits




2π =
α1 α2 = α1 + α2
H H =
= 2π
α1 α2 = α1 + α2
H H H = β0 β1 H β2 H β3
α1 α2
π α π = α
= H H H H H H
π π π
π = π/2 π/2 H π H −π/2 H π H
= 2π

Quantum circuits




2π =
α_1 α_2 = $\alpha_1 + \alpha_2$
H H =
$X(\alpha_1)$ H $X(\alpha_2)$ = β_0 $Z(\beta_1)$ $X(\beta_2)$ $Z(\beta_3)$
=

To go beyond

Polycontrolled PROPs.

Applications to reversible classical circuits.

Compilation of quantum circuits.

Thanks

(arXiv:2508.21756)

Diagrammatic Reasoning with Control as a Constructor, Applications to Quantum Circuits