# Quantum Circuit Completeness: Extensions and Simplifications

**32nd EACSL Annual Conference on Computer Science Logic 2024 (CSL'24)**

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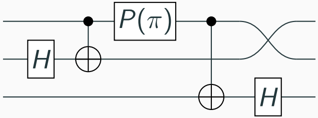
# What is it all about?

Quantum circuits are a rigourous graphical representation of quantum algorithms.

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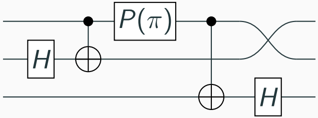


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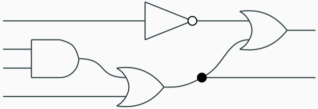

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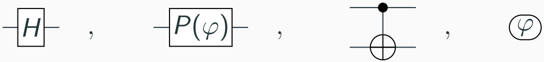


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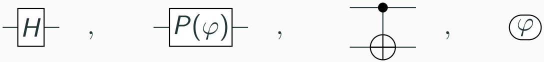


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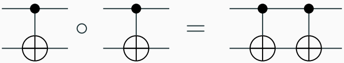

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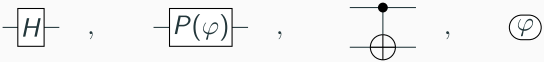


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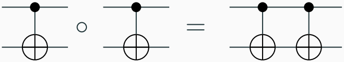

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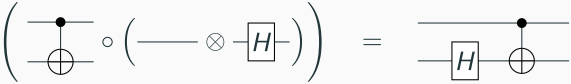

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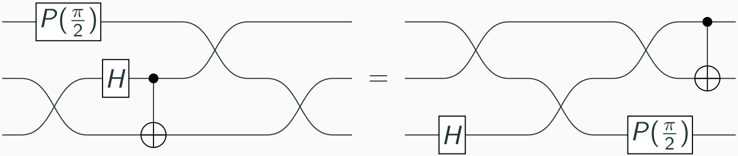

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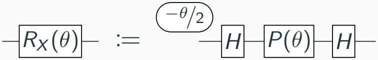


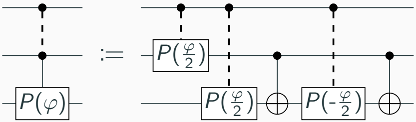

# Standard interpretation of quantum circuits

## Interpretation

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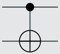
$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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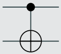
$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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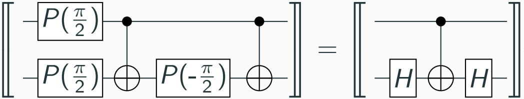
# Motivations

Distinct circuits can have the same interpretation.

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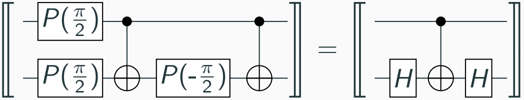


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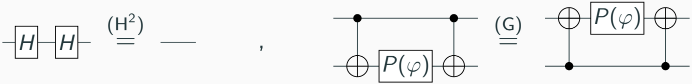


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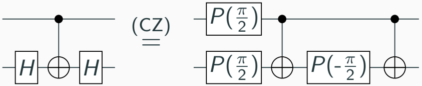


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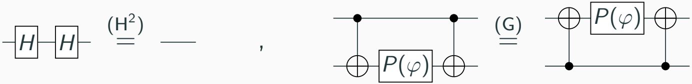


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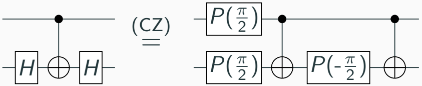


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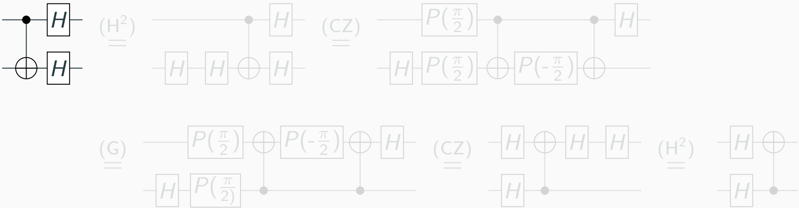

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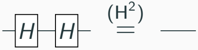


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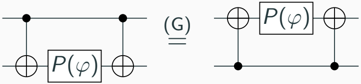


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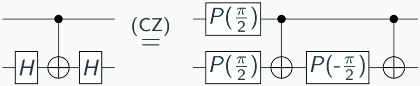


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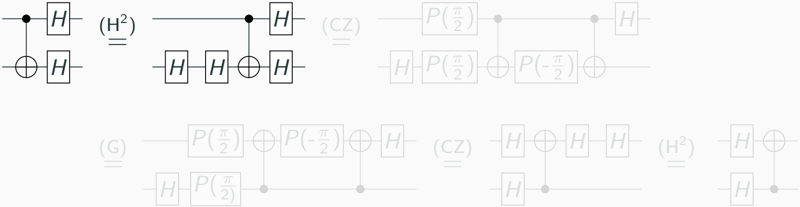

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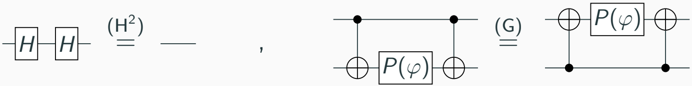


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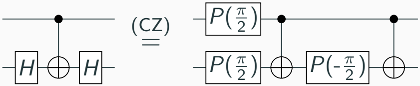


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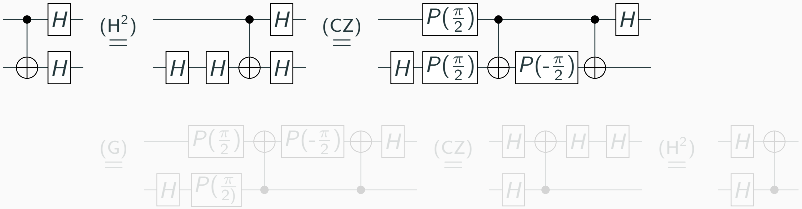

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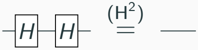


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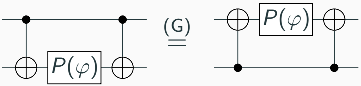


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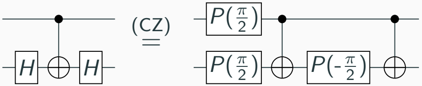


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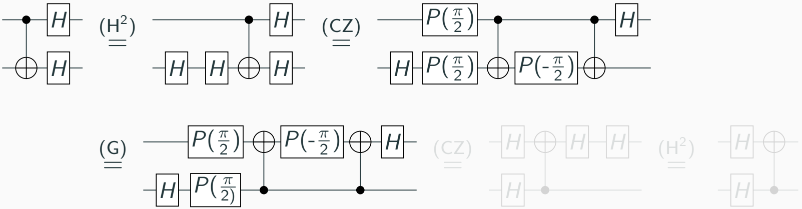

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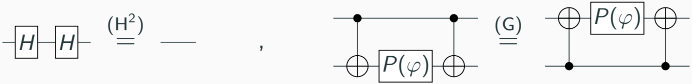


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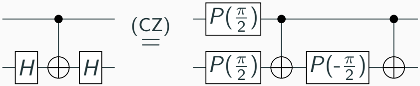


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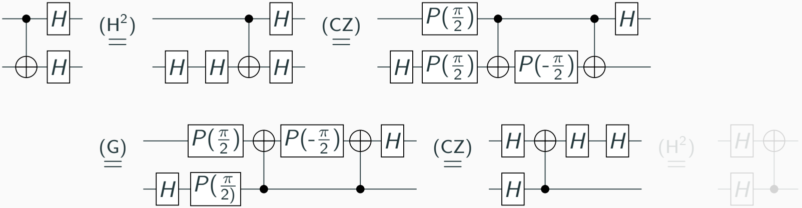

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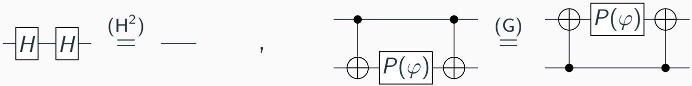


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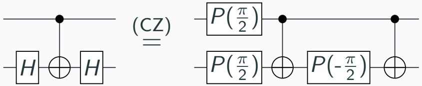


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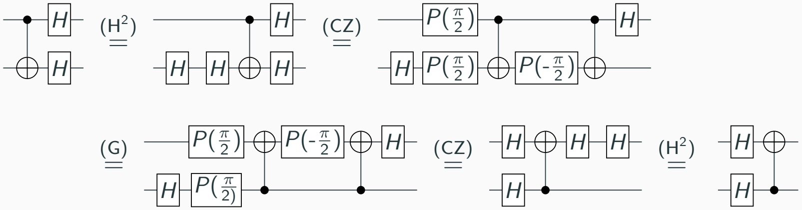

# Complete and sound equational theory

Is there an equational theory (i.e. a set of axioms) $\Gamma$ from which we can derive any true equation and only true equations?

**Soundness**

Any derivable equation is true.

$\forall C_1, C_2 \quad : \quad \Gamma \vdash C_1 = C_2 \implies [\![C_1]\!] = [\![C_2]\!]$

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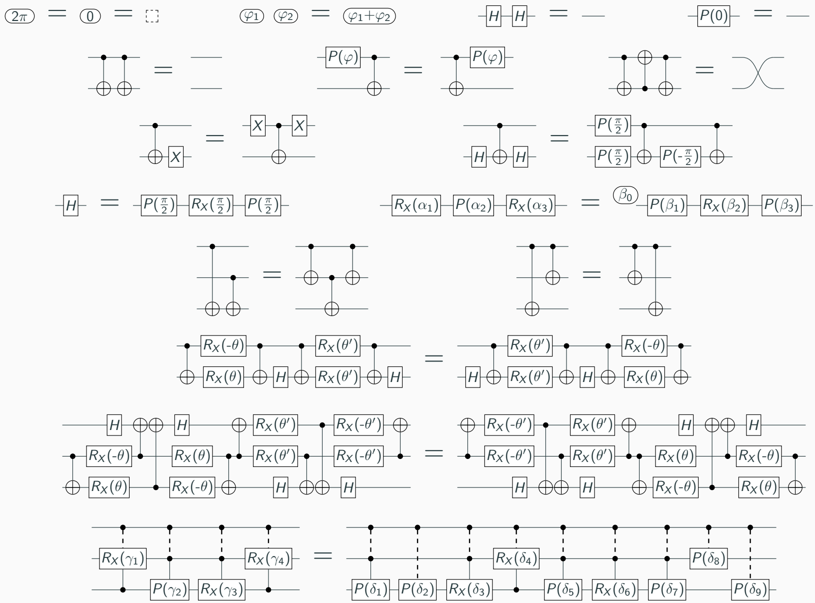

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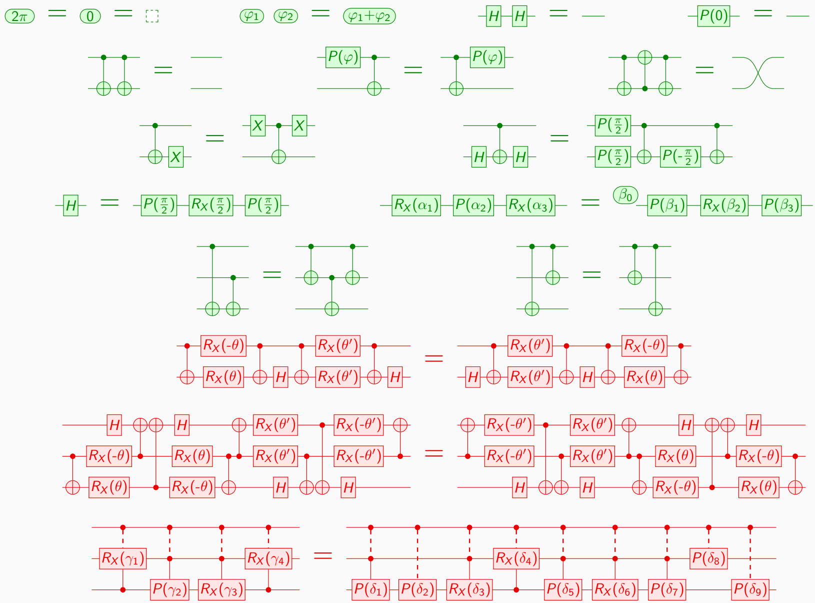
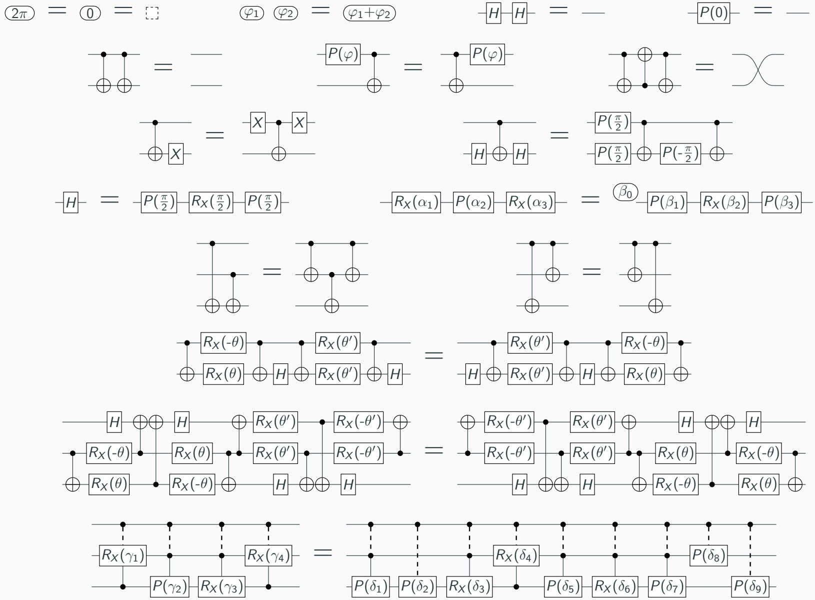


First contribution

# Simplification of the equational theory

# Simplifications



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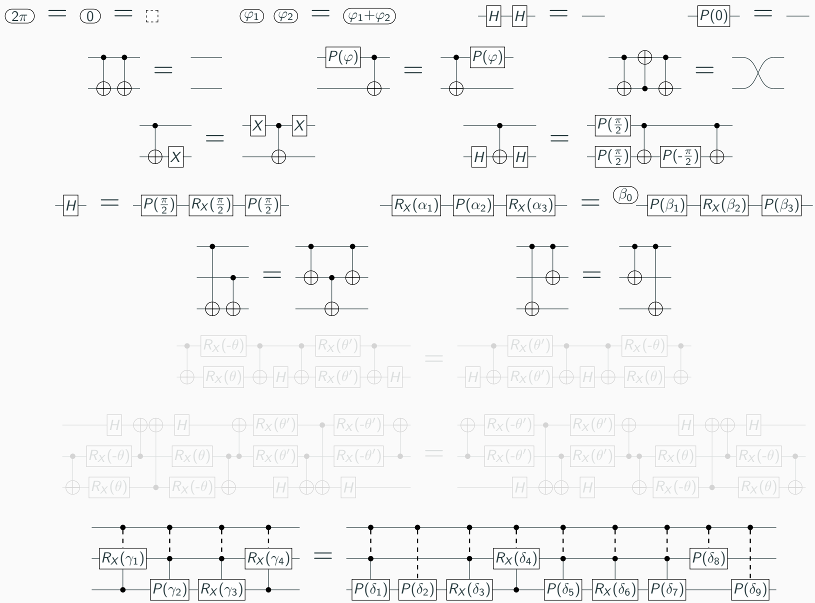

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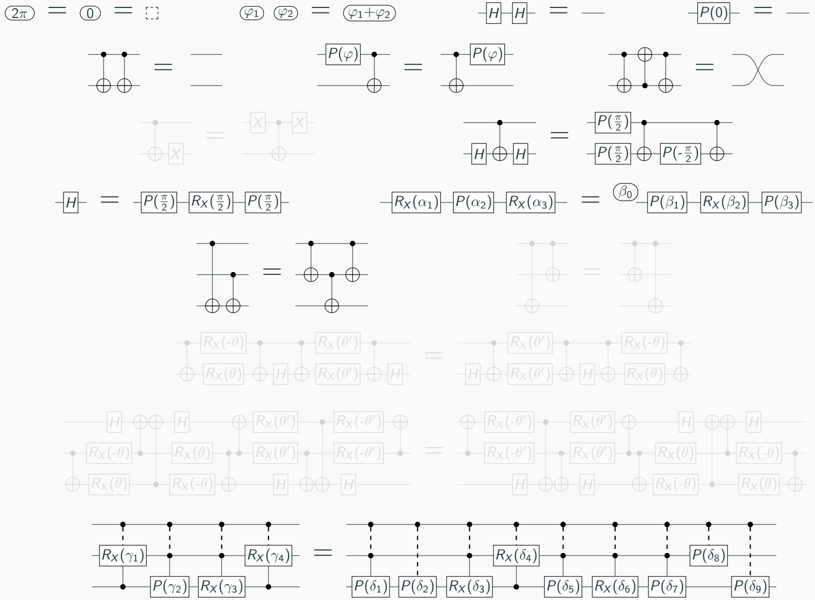

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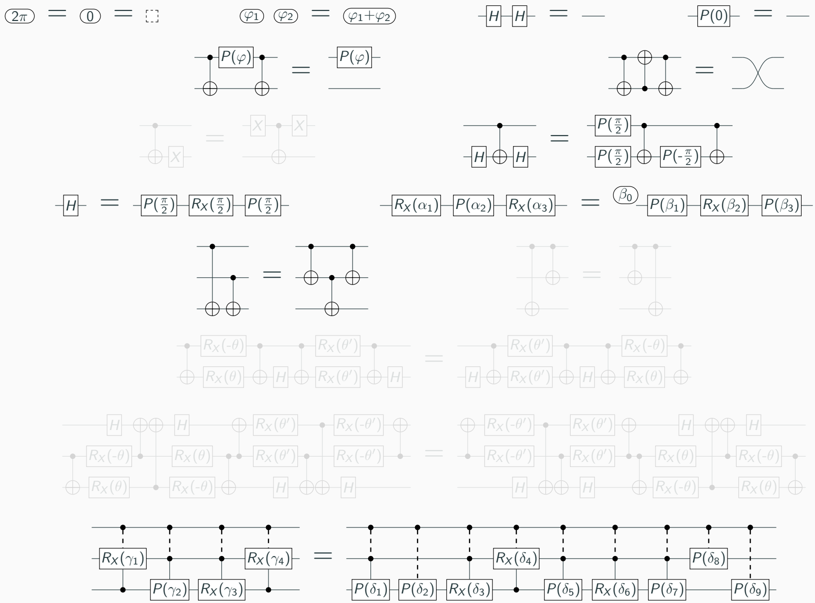

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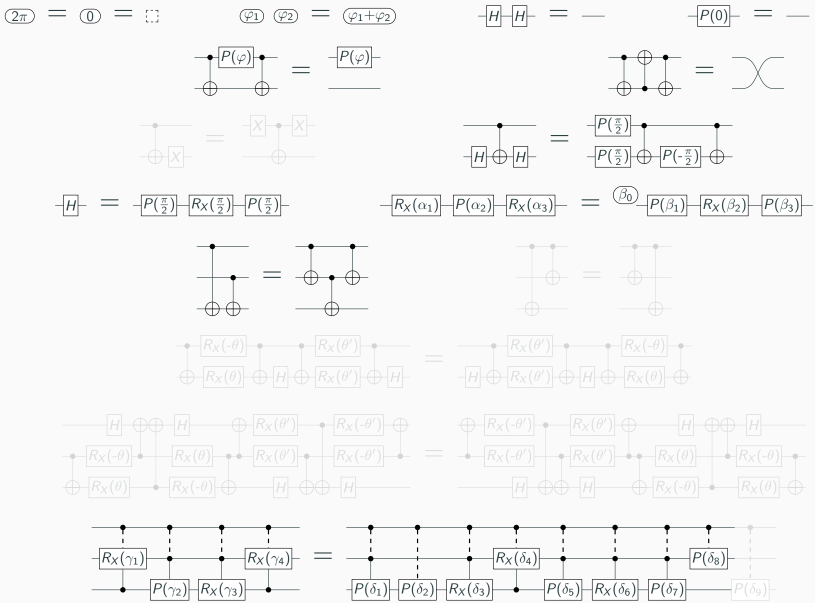
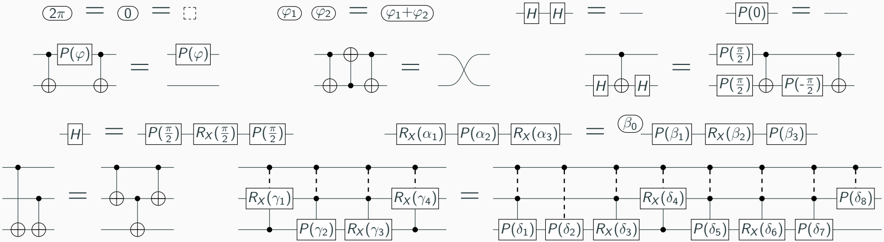


# Simplified complete and sound equational theory




**Theorem (Completeness)**

This equational theory is complete, i.e. any two equivalent circuits can be transformed into each other.

Second contribution

# Extension of the equational theory

# Extension to quantum circuits with ancillae

Quantum circuits with ancillae are generated by


together with

$\vdash$ and $\dashv$

respectively denoting qubit initialisation and qubit destruction.

(The generator $\dashv$ can only be applied to qubits in the $|0\rangle$-state.)

**Semantics**

We extend $[\![\cdot]\!]$ with $[\![\vdash]\!] = |0\rangle$ and $[\![\dashv]\!] = \langle 0|$.

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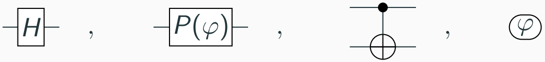


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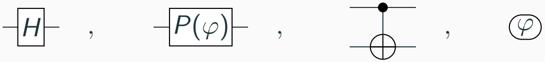


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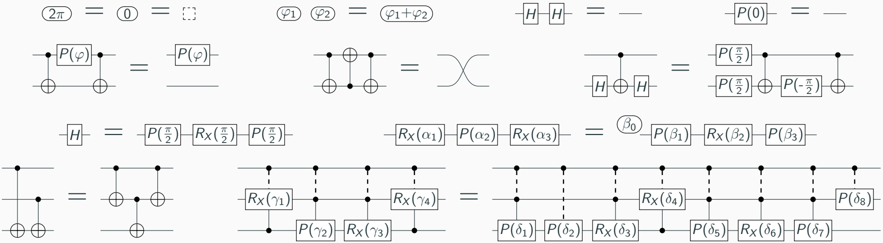
# Equational theory for quantum circuits with ancillae

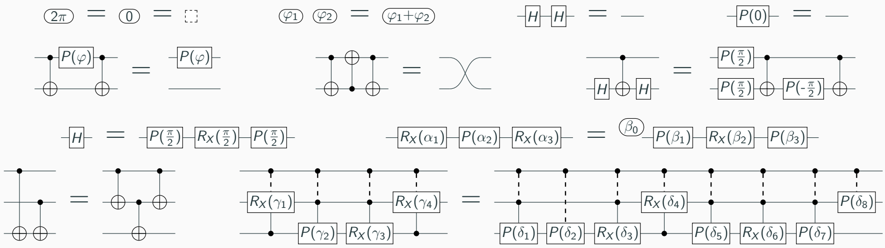


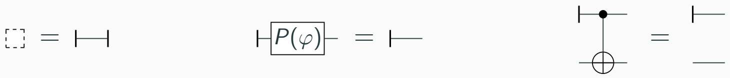


**Theorem (Completeness)**

Adding those three equations makes the equational theory complete for quantum circuits with ancillae.

# Equational theory for quantum circuits with ancillae

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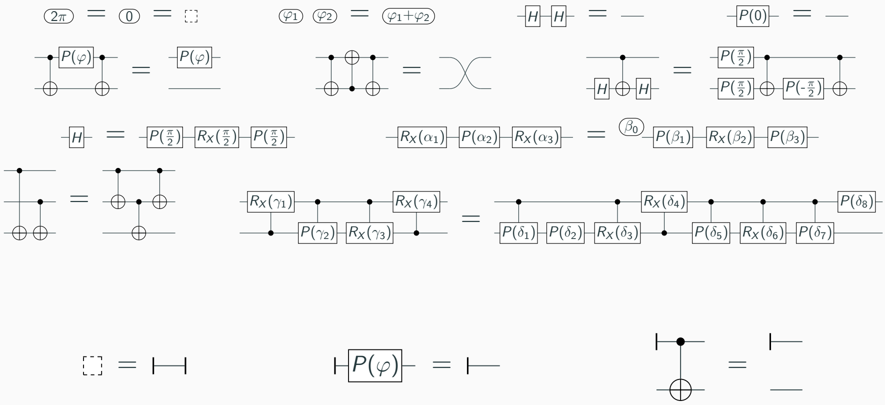
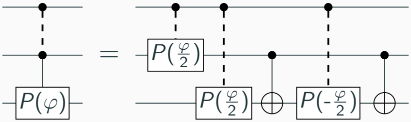


**Proposition**

The big rule can be replaced by its 2-qubits version, leading to an equational theory acting on a bounded number of qubits.

# Alternative definition of multi-controlled gates

Mutli-controlled gates are defined inductively




**Problem:** Cannot apply inductive hypothesis as angles are divided by 2.

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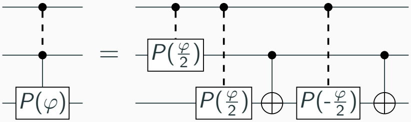


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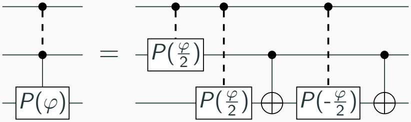


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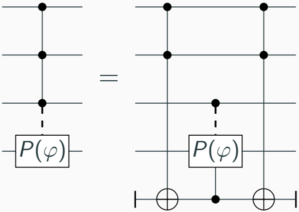

# Conclusion

- Simplification of the original equational theory, in particular removed two intricate rules.
- Introducing new techniques to reason on quantum circuits.
- Extension of the completeness result to circuits with ancillae.
- In these extended settings, the equational theory is made only of equations acting on at most 3 qubits.
- Other contribution: extension of the completeness result to circuits with discard (where any qubit can be discarded).

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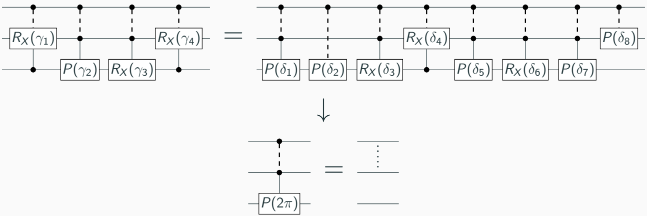
Replace the big rule by something simple.




Prove the minimality of the resulting equational theory.

**Theorem (Minimality)**

Each equation of the equational theory is necessary.

# Ongoing work [arXiv:2311.07476]

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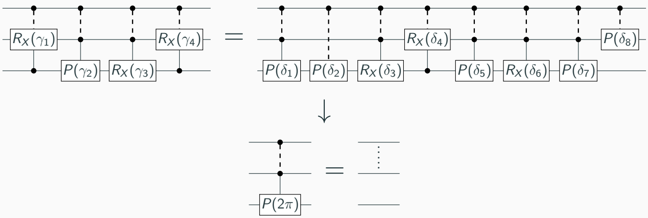


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# Thanks

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**Quantum Circuit Completeness: Extensions and Simplifications**

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